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Total Domination Number of Butterfly Graph

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ABSTRACT

Butterfly graphs are very important structures in computer architecture and communication techniques. Total Domination is an important parameter as it ensures connectivity in the network even after failure of few of the communication points. In this paper we find total domination number of BF(n) and deduce a relation between the domination number and total domination number of BF(n) as

γ_t **(BF(2))** $= 2^2 = \gamma$ **(BF(2)) + 2**

γ_t **(BF(3))** $= 2^3 = \gamma$ **(BF(3)) + 2**

γ_t **(BF(n))** $= \gamma$ **(BF(n))** **for n > 3**

Keywords— Butterfly graph, domination number, total domination number

I. INTRODUCTION

Berge [3] presents the problem of 5 queens as - Can we place 5 queens on a chess board such that every square on the board is covered by at least one queen? Another problem is that not only one or more queens dominate the squares, but each queen is dominated by another queen. The solution of this problem introduces the concept of total dominating sets in graphs.

Total Dominating sets were introduced by Cockayne, Dawes and Hedetniemi [4]. Some results on total domination can also be seen in Allan, Laskar and Hedetniemi [1]. In this paper we present the results on total domination number of butterfly graph BF(n).

Butterfly Graph BF(n) : The vertex set V of BF(n) is the set of ordered pairs (α; v) where $\alpha \in \{0, 1, 2, \ldots n-1\}$ and $v = x_{n-1}\, x_{n-2} \ldots\ldots x_1\, x_0$ is a binary string of length n where $x_i = 0$ or 1. There is an edge from a vertex (α; v) to a vertex (α'; v') where $\alpha' \equiv \alpha + 1 \pmod n$ and $x_j = x_j'\ \forall\ j \neq \alpha'$. A butterfly graph BF(n) is an n - partite graph with n levels. Each level L_k for $k = 0, 1, \ldots, n-1$ has 2^n vertices and $L_k = \{ (k; v) / v = x_{n-1}\, x_{n-2} \ldots\ldots x_1\, x_0,\ x_i = 0$ or $1\}$. Using decimal representation of the binary string we can write $L_k = \{ (k; m) /$ where $k = 0, 1, 2, \ldots n-1$ and $m = \sum x_j 2^j,\ j = 0,1, \ldots n-1) \}$.

II. SPECIAL RECURSIVE CONSTRUCTION OF BF(4k)

The recursive construction given by Barth and Raspaud [2] constructs BF(n) from 2 copies of BF(n-1), which gives that first 4 consecutive levels L_0, L_1, L_2, L_3 consist of 2^{n-3} copies of BF(4) in BF(n) for n > 4. Now we present a special recursive construction of BF(n) for n = 4k from BF(4), which is of great advantage for finding dominating sets of BF(4k) from the dominating set of BF(4).

Consider 2^4 copies of the graph BF(4(k-1)) in levels $L_0, L_1, L_2, \ldots\ldots L_{4k-5}$ and place them next to each other. Now call a set of 2^{n-3} vertices in each copy in these levels as a vertex group. Define a set of parallel edges between these vertices as an edge between vertex groups. Now the 4 levels L_{4k-4}, L_{4k-3}, L_{4k-2} and L_{4k-1} consist of 2^4 vertex groups in each level with edges as defined in BF(4) without winged edges. We shall call this a **pattern of BF(4),** without winged edges. Thus BF(4k) can be obtained recursively from BF(4(k-1)) by taking 2^4 copies of BF(4(k-1)), adding 4 levels of 2^4 vertex groups and defining the edges between these vertex groups in the same way as the edges between vertices of BF(4). Figure 2 illustrates this process for k = 2, namely for BF(8).

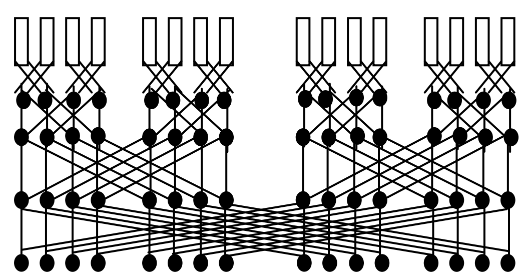

Figure 1: Special recursive construction of BF(8)

Indrani [6] has given minimal dominating sets of butterfly graphs and the domination numbers of BF(n) found are as follows.

$\gamma(BF(n)) = 2$ if $n = 2$
$= 6$ if $n = 3$
$= (2k+r)\, 2^{n-1}$ if $n = 4k+r$, $r = 0,1$ and $k \varepsilon Z^+$
$= (k+1)\, 2^n$ if $n= 4k+r$, $r = 2, 3$, $k \varepsilon Z^+$.

Because of the n-partite structure of BF(n), we choose vertices into the total dominating set from two consecutive levels say L_i and L_{i+1} which together dominate all the vertices of 4 levels L_{i-1}, L_i, L_{i+1} and L_{i+2}. But we require an additional condition, every vertex in the total dominating set T must be adjacent to some vertex of T.

III. MAIN RESULT

Theorem : The total domination number of Butterfly graph BF(n) is

$\gamma_t(BF(n)) = 2^2$ for $n = 2$
$= 2^3$ for $n = 3$
$= (2k+r)\, 2^{n-1}$ for $n = 4k+r$, $r= 0, 1$, and $k \varepsilon Z^+$
$= (k+1)\, 2^n$ for $n = 4k+r$, $r = 2,3$ and $k \varepsilon Z^+$.

The butterfly graphs BF(2) and BF(3) have a special structure. BF(2) is the only butterfly graph having parallel edges and BF(3) is the only butterfly graph having triangles. So we give the total domination number of these two graphs separately. We find total domination number of BF(4) which is a 4-regular graph without triangles and then using Special Recursive construction of BF(4k) the results for total domination number of BF(4) are extended for BF(4k). Later using the Recursive construction 1 we find total domination numbers of BF(4k+r) for r = 1, 2, 3.

Case 1 : n = 2.

Consider the graph BF(2). We know that $\gamma(BF(2)) = 2$. Without loss of generality, we assume that a dominating set of BF(2) be $D = \{(0; m_1), (1; m_2)\}$, where $m_1, m_2 \in \{0, 1, 2, 3\}$ and $m_1 + m_2 = 3$. Possible values for m_1 and m_2 are 0, 3 or 1, 2. From the definition of edges in BF(2) these pairs of vertices in D are not adjacent. Clearly D is not a total dominating set as the two vertices in D are not adjacent and hence does not dominate each other. To dominate the vertices of D, let us include a new vertex (k; t) into D. If k = 0 then (k; t) is not adjacent to $(0; m_1)$ and if k = 1 then (k; t) is not adjacent to $(1; m_2)$. So there is no single vertex (k; t) which dominates both $(0; m_1)$ and $(1; m_2)$ of D. Thus a dominating set of cardinality 3 is not a total dominating set. Hence if a total dominating set T exists in BF(2) then $|T| > 3$.

Now $(0; m_1)$ is adjacent to 3 vertices of level L_1, so we include any one of these 3 vertices into D say $(1; m_3)$ where $m_3 \neq m_2$. Again vertex $(1; m_2)$ is adjacent to 3 vertices of L_0, so we include any one of these three vertices into D say $(0; m_4)$ where $m_4 \neq m_1$. Thus the vertices $(1; m_3)$ and $(0; m_1)$ are adjacent and similarly the vertices $(1; m_2)$ and $(0; m_4)$ are adjacent .

$T=\{(0;m_1), (1;m_2), (1; m_3), (0; m_4) / m_1 \neq m_4 \text{ and } m_2 \neq m_3\}$.

Clearly $D \subseteq T$. So T is a dominating set. Also vertices $(1; m_3)$ and $(0; m_1)$ dominate each other and similarly the vertices $(1; m_2)$ and $(0; m_4)$ dominate each other, it follows that T is a total dominating set of minimum cardinality. Hence $\gamma_t(BF(2)) = 4 = \gamma(BF(2)) + 2$.

Two possible choices of total dominating sets are illustrated in the figure below :

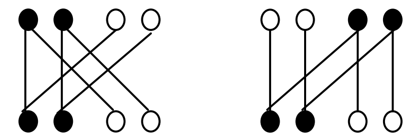

Figure 2: Two Total dominating sets for BF(2)

Case 2 : n = 3.

Consider the graph BF(3). We know that $\gamma(BF(3)) = 6$. As shown in [6] any dominating set D of cardinality 6 for BF(3) is an independent set. So D can not be a total dominating set. By recursive construction 1 we observe that BF(3) contains two copies of BF(2) and a level with 8 vertices. Let us consider the first copy of BF(2) along with the first four vertices of level L_2 as left part of BF(3) and the remaining part as right part of BF(3). A minimal dominating set D of BF(3) has 3 vertices, one vertex from each level, from left and right parts of BF(3). Hence, by symmetric structure of butterfly graph, we need to add at least one vertex each from left part and right part of BF(3) to D, in order to get a total dominating set. Thus $\gamma_t(BF(3)) \geq 8$. Now we present a construction of a total dominating set of cardinality 8 for BF(3). Consider a set of vertices from level L_k given by

$T_1=\{(k; m_1), (k; m_2) (k; m_3) (k; m_4) / |m_i - m_j| \neq 2^k, 2^{k+1}\}$.

Vertices in T_1 dominate all the vertices of L_{k-1} and L_{k+1} (Lemma 4.3 of [6]). Now to dominate the remaining vertices of $L_k \backslash T_1$, we select vertices from L_{k-1} or L_{k+1}. A vertex (k-1; t) of L_{k-1} is adjacent to two vertices (k; t) and (k; s) where $|t - s| = 2^k$. So only one of these vertices is in T_1 and the other is in $L_k \backslash T_1$. So (k-1; t) dominates one vertex of $L_k \backslash T_1$. Thus every vertex of L_{k-1} dominates one vertex of T_1 and one vertex of $L_k \backslash T_1$. So we need to select 4 vertices from L_{k-1} to dominate 4 vertices of $L_k \backslash T_1$.

Let $T_2 =\{(k-1;t_1), (k-1; t_2) (k-1; t_3) (k-1; t_4) / |t_i - t_j| \neq 2^k \}$.

Take $T = T_1 \cup T_2$. So $|T| = 4 + 4 = 8$. Now vertices of T_1 dominate all the vertices of L_{k-1} and L_{k+1} and vertices of T_2 dominate all the vertices of $L_k \backslash T_1$. So vertices of T dominate all the vertices of BF(3). As vertices of T_1 are adjacent to vertices of T_2, it follows that T is a total dominating set of BF(3). As $\gamma_t(BF(3)) \geq 8$, T is a minimal total dominating set of BF(3).

Thus $\gamma_t(BF(3)) = 8 = \gamma(BF(3)) + 2$

Case 3 : n = 4

Consider the graph BF(4). We know that $\gamma(BF(4)) =16$. As every total dominating set is dominating set, $\gamma_t(BF(4)) \geq 16$. Consider a dominating set of BF(4) given by $D = D_1 \cup D_2$, where the two sets D_1 and D_2 are

$D_1 = \{ (1; m_1), (1; m_2) \ldots\ldots (1; m_8) / |m_i - m_j| \neq 2, 4\}$ and $D_2 = \{ (2; s_1), (2; s_2) \ldots\ldots (2; s_8) / |s_i - s_j| \neq 4\}$.

Clearly from the definition of adjacency of vertices in two consecutive levels, the vertices in D_1 dominate all the vertices of level L_0 and L_2. As vertices of

D_2 belong to L_2, it is clear that a vertex in D_1 dominates a vertex in D_2 and vice-a-versa. So this set D dominates all the vertices of BF(4) and hence D is a total dominating set of BF(4). Since $|D| = 16$, D is a minimum total dominating set of BF(4). Therefore $\gamma_t(BF(4)) = 16$.

Case 4 : n = 4k.

We prove the result for BF(4k) by using the Principle of Mathematical Induction.

Step 1 : Let k = 1. So n = 4. For k = 1, from theorem 3, the result is true for BF(4).

Step 2 : Let us assume that the result is true for k = t. We prove the result for k = t+1. Consider the graph BF(4(t+1)). From Special Recursive Construction of BF(4k) , BF(4t+4) is decomposed into 16 copies of BF(4t) without wings and the last 4 levels form a pattern with 2^4 vertex groups, where each vertex group has 2^{4t} vertices. From the induction hypothesis the result is true for BF(4t). Consider a total dominating set S_i of cardinality t x 2^{4t} of BF(4t). The last 4 levels of BF(4t+4) form a pattern of BF(4) which is isomorphic to BF(4) without wings. Consider a total dominating set T_1 with 16 vertex groups for this pattern of BF(4) formed between the last 4 levels where each vertex group has 2^{4t} vertices.

Suppose we get a total dominating set of cardinality less than 16 x 2^{4t} for the pattern of BF(4). Since pattern of BF(4) is isomorphic to the graph BF(4) without wings, this assumption gives that we can get a total dominating set for BF(4) of cardinality less than 2^4, which is a contradiction as $\gamma_t(BF(4)) = 2^4$.

Let $T = \left(\bigcup_{i=1}^{16} S_i\right) \cup T_1$.

For i = 1, 2 … 16, each S_i is a minimum total dominating set of BF(4t) between levels L_0 to L_{4t-1} and T_1 is a minimum total dominating set of pattern of BF(4) between L_{4t} to L_{4t-3}. Thus all the S_i's and T_1 are disjoint sets. Hence T is a minimum total dominating set of BF(4t+4) whose cardinality is $|T| = 16 \text{ x } t \text{ x } 2^{4t} + 16 \text{ x } 2^{4t} = t \text{ x } 2^{4t+4} + 2^{4t+4} = (t+1)\, 2^{4(t+1)}$.

So the result is true for k = t+1. Hence by the Principle of Mathematical Induction the result is true for all positive integers k. Thus $\gamma_t(BF(4k)) = k \text{ x } 2^{4k}$.

Case 5 : n = (4k + r) r = 1, 2, 3

By Recursive Construction 1, there are 2^r copies of BF(4k) and r levels with 2^{4k+r} vertices in BF(4k+r). To get a minimum total dominating set for BF(4k+r), first we include the vertices in a minimum total dominating set of disjoint copies of BF(4k) in first 4k levels. Now for total domination of vertices, we include vertices into T from last r levels as follows :

For r = 1, we choose 2^{n-1} vertices from level L_{4k-1} to dominate all vertices of the last level L_{4k}.

For r = 2, we choose 2^{n-1} vertices from levels L_{4k-1} and L_{4k} each to dominate all vertices of the last two levels L_{4k} and L_{4k+1}.

For r = 3, we choose 2^{n-1} vertices from levels L_{4k} and L_{4k+1} each to dominate all vertices of the last three levels L_{4k} , L_{4k+1} and L_{4k+2}.

These vertices together with the minimum total dominating sets of r copies of BF(4k) form a minimum total dominating set of BF(4k+r). Combining these cases we get that the cardinality of minimum total dominating set for BF(4k+r) , |D| =

$= 2\gamma(BF(4k)) + 2^{4k} = 2\, k\, 2^{4k} + 2^{4k} = (2k+1)\, 2^{4k}$ if r=1

$= 2^2\gamma(BF(4k)) + 2.2^{4k+1} = 2^2 k 2^{4k} + 2.2^{4k+1} = (k+1) 2^{4k+2}$ if r=2

$= 2^3\gamma(BF(4k)) + 2.2^{4k+2} = 2^3 k 2^{4k} + 2.2^{4k+2} = (k+1) 2^{4k+3}$ if r=3

Thus we get

$$\gamma_t(BF(4k+r)) = (2k+r)\, 2^{n-1} \text{ for } r = 0,1 \text{ and } k \,\varepsilon\, Z^+$$
$$= (k+1)\, 2^n \quad \text{for } r = 2,3 \text{ and } k \,\varepsilon\, Z^+.$$

One such choice is illustrated in the figure below for the graph BF(5) in which bold lines show domination of the vertices from dominating set D and thin lines show V\ D vertices domination.

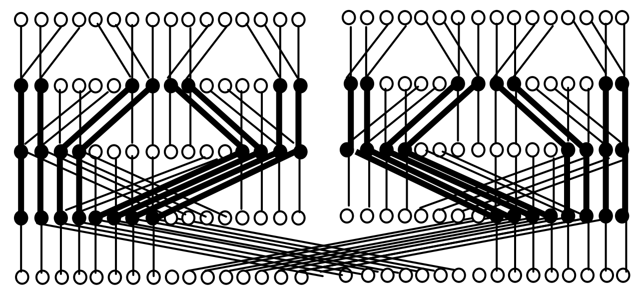

Figure 3: A Total dominating set for BF(5)

IV. CONCLUSION

The total domination number of BF(2) is 2^2 and BF(3) is 2^3. For n > 3 the total domination number is same as the domination number of BF(n). But every minimal dominating set is not minimal total dominating set. Only particular choices, form minimal total dominating sets. Thus $\gamma_t(BF(n)) = \gamma(BF(n))$ for n > 3.

REFERENCES

[1] Allan, R.B. Laskar, R Hedetniemi, S.T., A note on Total Domination, Discrete Maths 49, (1984), pp 7-13.

[2] Barth D. and Raspaud A., Two edge - disjoint Hamiltonian Cycles in the Butterfly Graphs, Information processing letters, 51(1994), 175–179.

[3] Berge, C., Graphs and Hypergraphs, North Holland Amsterdam (1973).

[4] Cockayne, E J., Dawes, R.M. Hedetniemi, S.T., Total Domination in Graphs, Networks, Vol. 10, (1980), pp 211-219

[5] Haynes, T.W. Hedetniemi, S.T. Slater, P.J., Fundamentals of Domination in graphs, Marcel Dekker Inc. New York, (1998).

[6] Indrani Kelker., B Maheswari, Domination Number of Butterfly graphs, Journal of Pure & Applied Physics, Vol. 21, No.3, 2009, pp 401-407.