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Note on Equitable Signed Graphs

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Abstract

In this paper, we define the equitable signed graph of a given signed graph and offer a structural characterization of equitable signed graphs. In the sequel, we also obtained switching equivalence characterization: $\Sigma \sim \mathcal{E}_t(\Sigma)$, where Σ and $\mathcal{E}_t(\Sigma)$ are signed graph and equitable signed graph respectively.

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1 Introduction

For standard terminology and notation in graph theory we refer Harary [6] and Zaslavsky [31] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Signed graphs, in which the edges of a graph are labelled positive or negative, have developed many applications and a flourishing literature (see [31]) since their first introduction by Harary in 1953 [7]. Their natural extension to multisigned graphs, in which each edge gets an n-tuple of signs—that is, the sign group is replaced by a direct product of sign groups—has received slight attention, but the further extension to gain graphs (also known as voltage graphs), which have edge labels from an arbitrary group such that reversing the edge orientation inverts the label, have been well studied [31]. Note that in a multisigned group every element is its own inverse, so the question of edge reversal does not arise with multisigned graphs.

A *signed graph* $\Sigma = (\Gamma, \sigma)$ is a graph $\Gamma = (V, E)$ together with a function $\sigma : E \to \{+, -\}$, which associates each edge with the sign + or -. In such a signed graph, a subset A of $E(\Gamma)$ is said to be *positive* if it contains an even number of negative edges, otherwise is said to be *negative*. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced* [7] if in every cycle the product of the edge signs is positive. Σ is antibalanced [8] if in every even (odd) cycle the product of the edge signs is positive (resp., negative); equivalently, the negated signed graph $-\Sigma = (\Gamma, -\sigma)$ is balanced. A *marking* of Σ is a function $\mu : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking μ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\mu(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking μ of Σ is called *canonical marking* of Σ. In a signed graph $\Sigma = (\Gamma, \sigma)$, for any $A \subseteq E(\Gamma)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [7]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$. (Sampathkumar [13]).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.1. For more new notions on signed graphs refer the papers ([10, 11, 14, 15], [17]-[27]).

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [31].

If $\mu : V(\Gamma) \to \{+, -\}$ is *switching function*, then *switching* of the signed graph $\Sigma = (\Gamma, \sigma)$ by μ means changing σ to σ^μ defined by:

$$\sigma^\mu = \mu(u)\sigma(uv)\mu(v).$$

The signed graph obtained in this way is denoted by Σ^μ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *isomorphic*, written as $\Sigma_1 \cong \Sigma_2$ if there exists a graph isomorphism $f : \Gamma_1 \to \Gamma_2$ (that is a bijection $f : V(\Gamma_1) \to V(\Gamma_2)$ such that if uv is an edge in Γ_1 then $f(u)f(v)$ is an edge in Γ_2) such that for any edge $e \in E(\Gamma_1)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $\Sigma_1 = (\Gamma_1, \sigma_1)$ *switches* to a signed graph $\Sigma_2 = (\Gamma_2, \sigma_2)$ (or that Σ_1 and Σ_2 are *switching equivalent*) written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $\Sigma_1^\mu \cong \Sigma_2$. Note that $\Sigma_1 \sim \Sigma_2$ implies that $\Gamma_1 \cong \Gamma_2$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [28]) or *cycle isomorphic* (see [30]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (See [30]):

Theorem 1.2. (**T. Zaslavsky** [30]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [16], the authors introduced the switching and cycle isomorphism for signed digraphs.

2 Equitable Signed Graphs

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In

1862, de Jaenisch [4] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [12] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [2] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability", which is now known as the domination number of a graph. Oystein Ore [9] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [29]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [3] published a survey paper, in which the notation $\gamma(\Gamma)$ was first used for the domination number of a graph Γ. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(\Gamma)$ and $|deg(u) - deg(v)| \leq 1$. Further, a vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$.

Let $u \in V(\Gamma)$. Then the number of vertices which are degree equitable with u, is called degree equitable number of u.

In [5], Dharmalingam introduced equitable graph of a graph as follows: Let $\Gamma = (V, E)$ be a graph. The equitable graph $\mathcal{E}_t(\Gamma)$ of Γ is defined as the graph with vertex set as $V(\Gamma)$ and two vertices u and v are adjacent if and only if u and v are degree equitable.

Motivated by the existing definition of complement of a signed graph, we extend the notion of equitable graphs to signed graphs as follows: The *equitable signed graph* $\mathcal{E}_t(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $\mathcal{E}_t(\Gamma)$ and sign of any edge uv in $\mathcal{E}_t(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called equitable signed graph, if $\Sigma \cong \mathcal{E}_t(\Sigma')$ for some signed graph Σ'. In the following section, we shall present a characterization of equitable signed graphs. The purpose of this paper is to initiate a study of this notion.

We now gives a straightforward, yet interesting, property of equitable signed graphs.

Theorem 2.1. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its equitable signed graph $\mathcal{E}_t(\Sigma)$ is balanced.*

Proof. Since sign of any edge uv in $\mathcal{E}_t(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ, by Theorem 1.1, $\mathcal{E}_t(\Sigma)$ is balanced. □

For any positive integer k, the k^{th} iterated equitable signed graph $\mathcal{E}_t(\Sigma)$ of Σ is defined as follows:

$$\mathcal{E}_t^0(\Sigma) = \Sigma,\ \mathcal{E}_t^0(\Sigma) = \mathcal{E}_t(\mathcal{E}_t^{k-1}(\Sigma))$$

Corollary 2.2. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and any positive integer k, $\mathcal{E}_t^k(\Sigma)$ is balanced.*

In [5], Dharmalingam characterized graphs for which $\mathcal{E}_t(\Gamma) \cong \Gamma$.

Theorem 2.3. (**K. M. Dharmalingam [5]**)
For any graph $\Gamma = (V, E)$, $\mathcal{E}_t(\Gamma) \cong \Gamma$ if and only if Γ is K_n.

We now characterize signed graphs which are switching equivalent to their equitable signed graphs.

Theorem 2.4. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\Sigma \sim \mathcal{E}_t(\Sigma)$ if and only if Γ is K_n and Σ is balanced.*

Proof. Suppose $\Sigma \sim \mathcal{E}_t(\Sigma)$. This implies, $\Gamma \cong \mathcal{E}_t(\Gamma)$ and hence by Theorem 2.3, we see that Γ must be isomorphic to K_n. Now, if Σ is any signed graph with underlying graph as K_n, Theorem 2.1 implies that $\mathcal{E}_t(\Sigma)$ is balanced and hence if Σ is unbalanced its $\mathcal{E}_t(\Sigma)$ being balanced cannot be switching equivalent to Σ in accordance with Theorem 1.2. Therefore, Σ must be balanced.

Conversely, suppose that Γ is K_n and Σ is balanced. Now, if Σ is a balanced signed graph with $\Gamma \cong K_n$, by Theorem 2.1, $\mathcal{E}_t(\Sigma)$ are balanced and hence, the result follows from Theorem 1.2. □

Theorem 2.5. *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their equitable signed graphs are switching equivalent.*

Proof. Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.1, $\mathcal{E}_t(\Sigma_1)$ and $\mathcal{E}_t(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [8] as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

Theorem 2.4 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 2.6. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\Sigma \sim \mathcal{E}_t(\eta(\Sigma))$ if and only if Γ is K_n and Σ is balanced.*

Corollary 2.7. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{E}_t(\Sigma) \sim \mathcal{E}_t(\eta(\Sigma))$.*

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $\mathcal{E}_t(\Sigma)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation of $\mathcal{E}_t(\Sigma)$ is balanced.

Theorem 2.8. *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $\mathcal{E}_t(\Gamma)$ is bipartite then $\eta(\mathcal{E}_t(\Sigma))$ is balanced.*

Proof. Since, by Theorem 2.1, $\mathcal{E}_t(\Sigma)$ is balanced, each cycle C in $\mathcal{E}_t(\Sigma)$ contains even number of negative edges. Also, since $\mathcal{E}_t(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{E}_t(\Sigma)$ is also even. Hence $\eta(\mathcal{E}_t(\Sigma))$ is balanced. □

3 Characterization of Equitable Signed Graphs

The following result characterize signed graphs which are equitable signed graphs.

Theorem 3.1. *A signed graph $\Sigma = (\Gamma, \sigma)$ is an equitable signed graph if and only if Σ is balanced signed graph and its underlying graph Γ is $\mathcal{E}_t(\Gamma)$.*

Proof. Suppose that Σ is balanced and its underlying graph Γ is an equitable graph. Then there exists a graph Γ' such that $\mathcal{E}_t(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1.1, there exists a marking μ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $\mathcal{E}_t(\Sigma') \cong \Sigma$. Hence Σ is an equitable signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is an equitable signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $\mathcal{E}_t(\Sigma') \cong \Sigma$. Hence by Theorem 2.1, Σ is balanced. □

Problem 3.2. *Characterize signed graphs for which:*

1. $\eta(\Sigma) \sim \mathcal{E}_t(\Sigma)$
2. $\Sigma \sim \eta(\mathcal{E}_t(\Sigma))$
3. $\eta(\Sigma) \cong \mathcal{E}_t(\Sigma)$
4. $\Sigma \cong \eta(\mathcal{E}_t(\Sigma))$
5. $\Sigma \cong \mathcal{E}_t(\Sigma)$.

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