HAMILTONIAN LACEABILITY IN CONE PRODUCT GRAPHS

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Abstract

A connected graph G is said to be Hamiltonian-*t*-laceable if there exists a Hamiltonian path between every pair of distinct vertices at a distance '*t*' in G and Hamiltonian-*t**-laceable if there exist at least one such pair, where t is a positive integer. In this paper we explore Hamiltonian- *t**- Laceability properties of the Cone product $C_p(n)$, Ring product *R(2n, 2n, 1)* and the C_g –product $C_g(n, mk)$ graphs, where $m \geq 2$ and *n, k* are positive integers.

Keywords: Hamiltonian-t^*-laceable graph, Cyclic product, Cone product.
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1. INTRODUCTION

Let *G* be a finite, simple connected undirected graph. Let *u* and *v* be two vertices in *G*. The distance between *u* and *v* denoted by *d(u,v)* is the length of a shortest *u-v* path in *G*. *G* is *Hamiltonian-t-laceable* if there exists a Hamiltonian path between every pair of vertices *u* and *v* with *d(u,v)=t* and *Hamiltonian-t*-laceable* if there exists at least one such pair with *d(u,v)=t* where t is a positive integer such that $1\leq t \leq$ diam*G*. The concept of Hamiltonian laceability of brick products of even cycles was studied by B. Alspach, C.C. Chen and Kevin Mc Avaney in [1]. In [2], Leena Shenoy and R. Murali have discussed the Hamiltonian laceability of Cyclic product C(2*n*,*m*). Using this concept, In this paper we explore Hamiltonian-t*-laceability of Ring product R(2*n*, 2*n*, 1) of graph. Also we establish laceability properties of Cone product and C_g – product graphs.

2. THE CONE PRODUCT GRAPH

The *Cone product graph* denoted by $C_p(n)$, $n\geq 2$ is defined as follows.

First take Corona of two paths P_n and P_n i.e, $P_n \circ P_n = G_1$ and denote the vertex set of G_1 by $V = r_k = \{a_{k1}, a_{k2}, a_{k3} \ldots\ldots\ldots a_{kn}\}$ where $1\leq k \leq n$.

Join each point r_k to a root vertex a_{k0}.

Next, for each $1\leq k\leq n$-1 an edge (called hooking edge) between the vertices a_{nk} in r_k and to $a_{(k+1)1}$ in r_{k+1} is drawn for each $1 \leq k \leq n$-1.

Finally, for $k = n$ an edge is draw to join a vertex a_{kn} in r_k to a_{11} in r_1.

The cone product $C_p(n)$ is shown in figure 2.1.

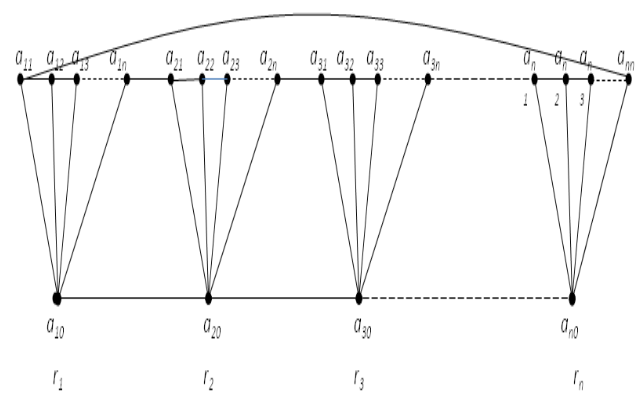

Fig- 2.1

Theorem 2.1. *The cone product $C_p(n)$, $n\geq 4$ is hamiltonian-t^*-laceable for $1\leq t \leq n$.*

Proof: Let G=$C_p(n)$ be a cone graph with vertex set $r_k = \{a_{k1}, a_{k2}, a_{k3} \ldots\ldots a_{kn}\}$ where $1\leq k \leq n$. The number of vertices in G is $n(n+1)=n^2+n$ and number of edges is (no. of vertices + $n^2 - 1$) = $2n^2$+n-1.

We consider the following cases

Case(i) For t=1

In G, $d(a_{11}, a_{10})$=1 and the path

P:{(a_{11}, a_{12})∪(a_{12}, a_{13})∪ (a_{13}, a_{14}) ∪ … … ∪ $(a_{1(n-1),} a_{1n})$}∪ (a_{1n}, a_{2n}) ∪ {$(a_{21,} a_{22})$ ∪ $a22,a23∪……..∪a2n-1,a2n∪a2n,a31∪a31,a32∪a32,a33∪………∪a3n-1,a3n∪a3n,a41∪……..∪akn-1,akn∪(ak0,a(k-1)0)∪(a(k-1)0,$ $a_{(k-2)0}$)∪ … … ∪ (a_{20}, a_{10}) is a Hamiltonian path

Hence C_p(n) is Hamiltonian- 1^*-laceable

Case(ii): For t=2

Clearly, $d(a_{11}, a_{20})$ =2 and the path

P:{(a_{11}, a_{10})∪(a_{10}, a_{12})∪ (a_{12}, a_{13}) ∪ (a_{13}, a_{14}) ∪ … … ∪ $(a_{1(n-1),} a_{1n})$}∪ (a_{1n}, a_{21}) ∪ {$(a_{21,} a_{22})$ ∪ $a22,a23∪(a23,a24)∪……..∪a2n-1,a2n∪a2n,a31∪a31,a32∪a32,a33∪………∪a3n-1,a3n∪a3n,a41∪……..∪akn-1,akn∪(ak0,a(k-1)0)∪(a(k-1)0,$ $a_{(k-2)0}$)∪ … … ∪ (a_{30}, a_{20}) is a Hamiltonian path.

Hence $C_p(n)$ is Hamiltonian- 2^*-laceable

Case(iii): For t=3

In G, $d(a_{11}, a_{30})$ =3 and the path

P:{(a_{11}, a_{10})∪(a_{10}, a_{12})∪ (a_{12}, a_{13}) ∪ (a_{13}, a_{14}) ∪ … … ∪ $(a_{1(n-1),} a_{1n})$}∪ (a_{1n}, a_{21}) ∪ {$(a_{21,} a_{22})$ ∪ $a22,a23∪(a23,a24)∪……..∪a2n-1,a2n∪a2n,a31∪a31,a32∪a32,a33∪………∪a3n-1,a3n∪a3n,a41∪……..∪akn-1,akn∪(ak0,a(k-1)0)∪(a(k-1)0,$ $a_{(k-2)0}$)∪ … … ∪ (a_{30}, a_{20}) is a Hamiltonian path.

Hence $C_p(n)$ is Hamiltonian- 3^*-laceable

Hence the Proof ■

Remark 2.1: If n =2 or 3, the Cone product $C_p(n)$ is Hamiltonian-2^*-laceable.

Figure 2.2, illustrates a Hamiltonian path from a_{11} to a_{20} in $C_p(3)$. This path is

P:{(a_{11}, a_{10})∪(a_{10}, a_{12}) ∪ (a_{12}, a_{13}) ∪ (a_{13}, a_{21}) ∪ (a_{21}, a_{22}) ∪ (a_{22}, a_{23}) ∪ (a_{23}, a_{31}) ∪ (a_{31}, a_{32}) ∪ (a_{32}, a_{33}) ∪ (a_{33}, a_{30}) ∪ (a_{30}, a_{20}).

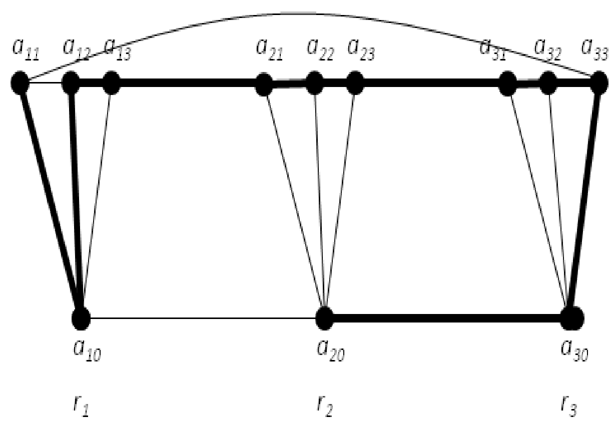

Fig- 2.2

3. LACEABILITY IN RING PRODUCT OF GRAPHS

First, we recall [2] the definition of Cyclic product graph.

Let m and n be positive integers. Let C_{2n} = $a_0, a_1, a_2\ a_3, a_4, a_5, ..., a_{2n-1}, a_0$ denote a cycle of order $2n$ (n>1). Then, the cyclic product of C_{2n} denoted by C($2n$, m) is defined as follows.

For m=1, C($2n$, 1) is obtained from C_{2n} by adding chords $a_k(a_{2n-k})$, 1≤ k ≤ (n-1) and $a_k(a_{2n})$, for k = n where the computation is performed under modulo $2n$.

Definition 3.1 : The Ring product R($2n$, $2n$, 1) is obtained by taking two copies of C($2n$,1) with vertex set $V_1 + V_2$ Where $V_1 = \{a_i\}$ and $V_2 = \{a_i^{'}\}$. Each vertex a_i in V_1 is joined by $a_i^{'}$ in V_2 , n≥ 3, $0 \le i \le 2n - 1$.

Example: Ring product R(6, 6, 1) is shown in Figure 3.1

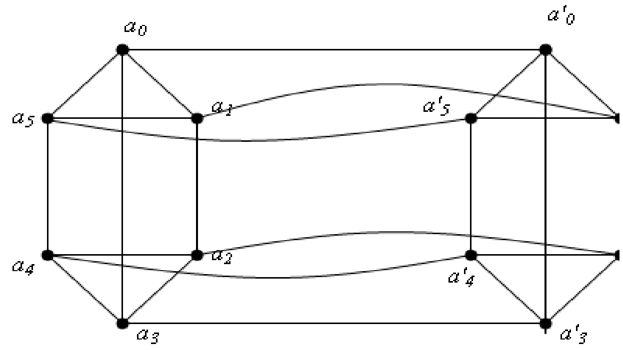

Fig-3.1

Theorem 3.1. *The graph R(2n,2n,1) is Hamiltonian-t^*-laceable for $1 \leq t \leq 3$.*

Proof: Consider the graph $G=R(2n,2n,1)$ with vertex set
V=$\{a_0, a_1, a_2, a_3, a_4 \ldots\ldots a_{2n-1}, a'_0, a'_1, a'_2, a'_3, a'_4, \ldots\ldots\ldots, a'_{2n-1}\}$.

G has $4n$ number of vertices and $3(n+1)$ number of edges.

We consider the following cases

Case(i): For t=1

In G, $d(a_0, a'_0)$=1 and the path P:$(a_0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_4) \cup \ldots\ldots\ldots \cup (a_{2n-1}, a'_{2n-1}) \cup (a'_{2n-1}, a'_{2n-2}) \cup (a'_{2n-2}, a'_{2n-3}) \cup (a'_{2n-3}, a'_{2n-4}) \cup \ldots\ldots\ldots \cup (a'_2, a'_1) \cup (a'_1, a'_0)$ is a Hamiltonian path.

Hence G is Hamiltonian-1^*-laceable

Case(ii): For t=2

Clearly, $d(a_0, a'_n) = 2$ and the path P:$(a_0, a_{2n-1}) \cup (a_{2n-1}, a_{2n-2}) \cup (a_{2n-2}, a_{2n-3}) \cup (a_{2n-3}, a_{2n-4}) \cup \ldots\ldots\ldots \cup (a_n, a_{n-1}) \cup (a_{n-1}, a_{n-2}) \cup (a_{n-2}, a_{n-3}) \cup \ldots\ldots\ldots. \cup (a_1, a'_1) \cup (a'_1, a'_{2n}) \cup (a'_{2n-1}, a'_{2n-2}) \cup (a'_{2n-2}, a'_2) \cup \ldots\ldots\ldots \cup (a'_2, a'_3) \cup (a'_3, a'_{2n-3}) \cup (a'_{2n-3}, a'_{2n-4}) \cup (a'_{2n-4}, a'_4) \cup (a'_4, a'_5) \cup (a'_{2n-5}, a'_{2n-6}) \cup (a'_{2n-6}, a'_6) \cup (a'_6, a'_7) \cup \ldots\ldots\ldots. \cup (a'_{n-1}, a'_n)$ is a Hamiltonian path.

Hence G is Hamiltonian-2^*-laceable

Case(ii): For t=3

In G, $d(a_0, a'_2) = 3$ and the path P:$(a_0, a_{2n-1}) \cup (a_{2n-1}, a_{2n-2}) \cup (a_{2n-2}, a_{2n-3}) \cup (a_{2n-3}, a_{2n-4}) \cup \ldots\ldots\ldots \cup (a_n, a_{n-1}) \cup (a_{n-1}, a_{n-2}) \cup (a_{n-3}, a_{n-4}) \cup \ldots\ldots\ldots. \cup (a_1, a'_1) \cup (a'_1, a'_{2n}) \cup (a'_{2n}, a'_{2n-1}) \cup (a'_{2n-1}, a'_{2n-2}) \cup (a'_{2n-2}, a'_{2n-3}) \cup \ldots\ldots\ldots \cup (a'_n, a'_{n-1}) \cup (a'_{n-1}, a'_{n-2}) \cup \ldots\ldots\ldots. \cup (a'_3, a'_2)$ is a Hamiltonian path.

Hence G is Hamiltonian-3^*-laceable. ■

Figure 3.2 shows Hamiltonian path in $G=R(10, 10, 1)$ between the vertices a_0 to a'_5 is shown. This path is P:$(a_0, a_9) \cup (a_9, a_8) \cup (a_8, a_7) \cup (a_7, a_6) \cup (a_6, a_5) \cup (a_5, a_4) \cup (a_4, a_3) \cup (a_3, a_2) \cup (a_2, a_1) \cup (a_1, a'_1) \cup (a'_1, a'_0) \cup (a'_0, a'_9) \cup (a'_9, a'_8) \cup (a'_8, a'_2) \cup (a'_2, a'_3) \cup (a'_3, a'_7) \cup (a'_7, a'_6) \cup (a'_6, a'_4) \cup (a'_4, a'_5)$

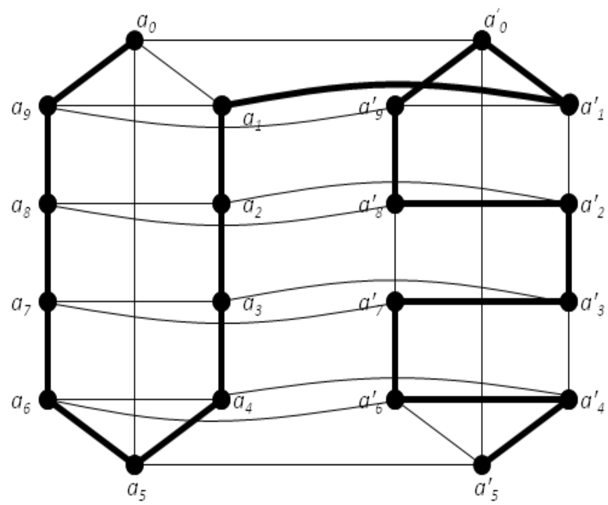

Fig- 3.2

4. C_g- PRODUCT

The C_g – Product $C_g(n, mk)$ is defined as follows:

Let $\{a_1, a_2, a_3, a_4 \ldots\ldots a_{n-1}, a_0 = a_n\}$ be n number of vertices and for each i, join an edge a_i to a_{i+mk}, where $m \geq 2$, and computation is performed under modulo n. Where $k = \left\lfloor \frac{n-2}{m} \right\rfloor$.

Example: The graph C_g (10,3*k*) is shown in the figure 4.1.

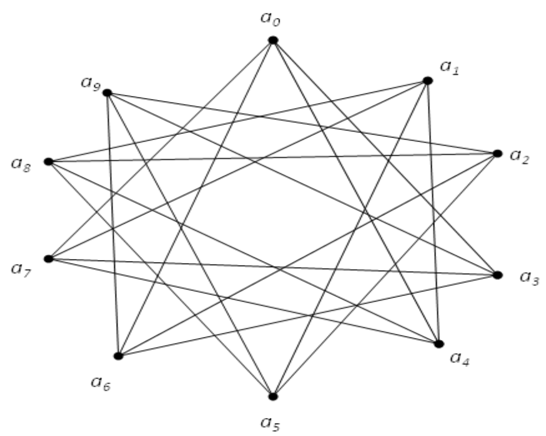

Fig- 4.1

Now, we consider the following theorem

Theorem 4.1 : *Let G=C_g (n, 2k), n≥ 8. Then, G is Hamiltonian- t^*-laceable for t=1,2. Where n≠3(l+2), l≥ 1*

Proof: Let G=C_g (n, 3k), n≥ 8. The vertex set of G is given by V={a_1, a_2, a_3, a_4, a_5……… a_{n-2}, a_{n-1}}

Case(i): For n= 3l+5, l≥ 1

Subcase(i): For t=1

In G, $d(a_1, a_4)$=1 and the path

P:(a_1, a_{n-1})∪ (a_{n-1}, a_{n-3})∪ (a_{n-3}, a_{n-5}) ∪ … … . .∪ (a_5, a_3) ∪(a_3, a_6)∪ … … … ∪(a_{n-2}, a_0)∪ (a_0, a_2) ∪ (a_2, a_4) is a Hamiltonian path.

Hence G is Hamiltonian- 1^*-laceable for n=3l+5.

Subcase(ii): For t=2

In G, $d(a_1, a_2)$=2 and the path

P:(a_1, a_{n-1})∪ (a_{n-1}, a_{n-3})∪ (a_{n-3}, a_{n-5}) ∪ … … . .∪ (a_3, a_0) ∪(a_0, a_{n-2})∪ (a_{n-2}, a_{n-4}) ∪ (a_{n-4}, a_{n-6}) ∪ … … … ∪ (a_4, a_2) is a Hamiltonian path.

Hence G is Hamiltonian- 2^*-laceable for n=3l+5.

Case(ii): For n= 3l+7, l≥ 1

Subcase(ii): For t=1

In G, d(a_1, a_4)=1 and the path

P:(a_1, a_{n-2})∪ (a_{n-2}, a_{n-5})∪ (a_{n-5}, a_{n-8}) ∪ … … . .∪ (a_5, a_2) ∪(a_2, a_{n-1})∪ (a_{n-1}, a_{n-4}) ∪ … … … ∪(a_6, a_6)∪ (a_3, a_0) ∪ (a_0, a_{n-3}) ∪ (a_{n-3}, a_{n-6}) ∪ … … . .∪ (a_7, a_4) is a Hamiltonian path.

Hence G is Hamiltonian- 1^*-laceable for n=3l+7.

Subcase(ii): For t=2

In G, $d(a_1, a_3)$=2 and the path

P:(a_1, a_{n-2})∪ (a_{n-2}, a_{n-5})∪ (a_{n-5}, a_{n-8}) ∪ … … . .∪ (a_5, a_2) ∪(a_2, a_{n-1})∪ (a_{n-1}, a_{n-4}) ∪ (a_{n-4}, a_{n-7}) ∪ … … … ∪(a_6, a_0)∪ (a_0, a_{n-3}) ∪ (a_{n-3}, a_{n-6}) ∪ … … . .∪ (a_7, a_3) is a Hamiltonian path.

Hence G is Hamiltonian- 2^*-laceable for n=3l+7.

Hence the proof. ■

In Figure 4.2 Hamiltonian path in G= C_y (8,3k), between the vertices $a_1$to a_3 is shown. This path is P:(a_1, a_7)∪ (a_7, a_5) ∪ (a_5, a_3)∪ (a_3, a_0) ∪ (a_0, a_6)∪ (a_6, a_4)∪ (a_4, a_2).

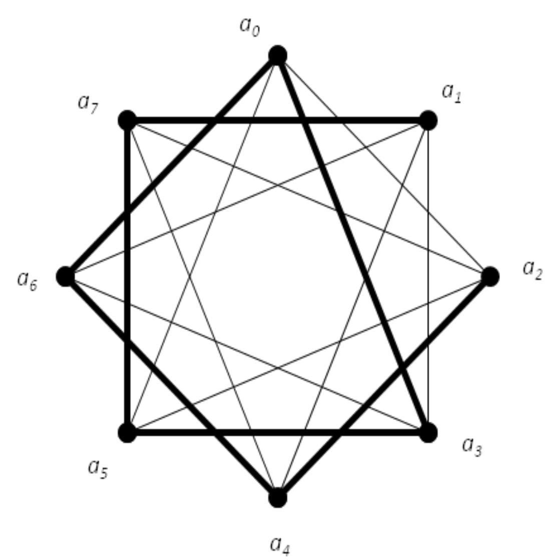

Fig- 4.2

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