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PROJECTIVE EQUIVALENCE BETWEEN TWO FAMILIES OF FINSLER METRICS

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Abstract. In this paper, we find the necessary and sufficient condition to characterize the projective relation between two subclasses of (α,β)-metrics $L = \alpha + 2\beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a manifold M with dimension $n > 2$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero 1-forms.

1. Introduction

In Finsler geometry, two Finsler metrics L and $\bar{L}$ on a manifold M are said to be projectively related if $G^i = \bar{G}^i + Py^i$, where G^i and $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$ respectively and $P = P(x,y)$ is a scalar function on the slit tangent bundle TM_0. In this case, any geodesic of the first is also geodesic for the second and viceversa. The projective changes between two Finsler spaces have been studied by [2], [3], [9], [10], [14], [15].

(α,β)-metrics form a special and very important classes of Finsler metrics which can be expressed in the form $L = \alpha\phi(s) : s = \frac{\beta}{\alpha}$, where α is a Riemannian metric, β is a 1-form and ϕ is a C^∞ positive function on the definite domain. In particular, when $\phi = 1/s$, the Finsler metric $L = \frac{\alpha^2}{\beta}$ is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina [5]. They together with Randers metric are C-reducible [8]. However, Randers metric are regular Finsler metric but Kropina metric is non-regular Finsler metric. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [4], [11]. Also, there are interesting applications in relativistic field theory, evolution and developmental biology.

Based on Stavrino's work on Finslerian structure of anisotropic gravitational field [12], we know that the anisotropy is an issue of the background radiation for all possible (α,β)-metrics. Then the 1-form β represents the same direction of the observed anisotropy of the microwave background radiation. That is, if two

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(α,β)-metrics $L=\alpha\phi(\frac{\beta}{\alpha})$ and $\bar{L}=\bar{\alpha}\phi(\frac{\bar{\beta}}{\bar{\alpha}})$ are the same anisotropy directions (or, they have the same axis rotation to their indicatrices), then their 1-form β and $\bar{\beta}$ are collinear, there is a function $\mu\in C^\infty(M)$ such that $\beta(x,y)=\mu\bar{\beta}(x,y)$. By [3], for the projective equivalence between a general (α,β)-metric and a Kropina metric, we have the following lemma

Lemma 1.1. *Let $L=\alpha\phi\left(\frac{\beta}{\alpha}\right)$ be an (α,β)-metric on n-dimensional manifold $M(n>2)$ satisfying that β is not parallel with respect to $\alpha, db\neq 0$ everywhere (or) $b=$ constant and L is not of Randers type. Let $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on the manifold M, where $\bar{\alpha}=\lambda(x)\alpha$ and $\bar{\beta}=\mu(x)\beta$. Then L is Projectively equivalent to $\bar{L}$ if and only if the following equations holds*

$$[1+(k_1+k_2s^2)s^2+k_3s^2]\phi'' = (k_1+k_2s^2)(\phi-s\phi'), \tag{1.1}$$

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}}+\theta y^i-\sigma(k_1\alpha^2+k_2\beta^2)b^i, \tag{1.2}$$

$$b_{i|j} = 2\sigma[(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j], \tag{1.3}$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i), \tag{1.4}$$

where $\sigma=\sigma(x)$ is a scalar function and $k_1,\ k_2$ and k_3 are constants. In this case both $L=\alpha\phi(\frac{\beta}{\alpha})$ and $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

The purpose of this paper is to study the projective equivalence between two families of Finsler metrics. The main results of the paper are as follows:

Theorem 1.2. *Let $L=\alpha+2\beta+\frac{\beta^2}{\alpha}$ be a (α,β)-metric and $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on a n-dimensional manifold $M(n>2)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then L is projectively equivalent to $\bar{L}$ if and only if they are Douglas metrics and the geodesic co-efficient of α and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha+2\alpha^2\tau b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i)+\theta y^i, \tag{1.5}$$

where $b^i=a^{ij}b_j,\ \bar{b}^i=\bar{a}^{ij}\bar{b}_j,\ \bar{b}^2=\|\bar{\beta}^2\|_{\bar{\alpha}}$, $\tau=\tau(x)$ is scaler function and $\theta=\theta_iy^i$ is a 1-form on M.

By [6] and [7], we obtain immediately from theorem (1.2), that

Proposition 1.3. *Let $L=\alpha+2\beta+\frac{\beta^2}{\alpha}$ be an (α,β)-metric and $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on a n-dimensional manifold $M(n>2)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if the following holds*

$$G^i_\alpha+2\alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i)+\theta y^i, \tag{1.6}$$

$$b_{i|j} = 2\tau\{(1+2b^2)a_{ij}-3b_ib_j\}, \tag{1.7}$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i), \tag{1.8}$$

where $b_{i|j}$ denote the coefficient of the covariant derivative of β with respect to α.

2. Preliminaries

We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [2]

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^i) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be projectively related if and only if their spray coefficients have the relation [2]

$$G^i = \bar{G}^i + P(y)y^i, \tag{2.2}$$

where $P(y)$ is a scalar function on $TM\backslash\{0\}$ and homogeneous of degree one in y.

For a given Finsler metric $L = L(x, y)$, the geodesics of L satisfy the following ODE:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}.$$

Let $\phi = \phi(s),\ |s| < b_0$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b < b_0). \tag{2.3}$$

If $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_iy^i$ is 1-form satisfying $||\beta_x||_\alpha < b_0$, $\forall x \in M$, then $F = \alpha\phi(s), s = \beta/\alpha$, is called an (regular) (α, β)-metric. In this case, the fundamental form of the metric tensor induced by L is positive definite.

Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be covariant derivative of β with respect to α.
Denote $r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})$ and $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$.
Note that β is closed if and only if $s_{ij} = 0$ [13]. Let $s_j = b^i s_{ij}$, $s^i_j = a^{il}s_{lj}$, $s_0 = s_iy^i$, $s^i_0 = s^i_jy^j$ and $r_{00} = r_{ij}y^iy^j$.

The relation between the geodesic coefficients G^i of L and geodesic coefficient G^i_α of α is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.4}$$

where

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\{(\phi - s\phi') + (b^2 - s^2)\phi''\}},$$

$$Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Psi = \frac{\phi''}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}.$$

For a Kropina metric $F = \frac{\alpha^2}{\beta}$, it is very easy to see that it is not a regular (α, β)-metric but the relation $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0$ is still true for $|s| > 0$.
In [6], the authors characterized the (α, β)-metrics of Douglas type.

Lemma 2.1. [6]*: Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be a regular (α, β)-metric on an n-dimensional manifold $M(n > 2)$. Assume that β is not parallel with respect to α and $db \neq 0$ every where or $b =$constant, and F is not of Randers type. Then F is a Douglas metric if and only if the function $\phi = \phi(s)$ with $\phi(0) = 1$ satisfies following*

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{2.5}$$

and β satisfies

$$b_{i|j} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j], \tag{2.6}$$

where $b^2 = \|\beta\|^2_\alpha$ and $\sigma = \sigma(x)$ is a scalar function and $k_1,\ k_2$ and k_3 are constants with $(k_2, k_3) \neq (0, 0)$.

For a Kropina metric, we have the following

Lemma 2.2. [7]*: let $L = \frac{\alpha^2}{\beta}$ be Kropina metric on an n-dimensional manifold M. Then*
(i) $(n \geq 3)$ Kropina metric L with $b^2 \neq 0$ is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_j s_i). \tag{2.7}$$

(ii) $(n = 2)$ Kropina metric L is a Douglas metric.

Definition 2.3. [2]: Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.8}$$

where G^i are the spray coefficients of L. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes from (2.8). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{L}$.
Now, first we compute the Douglas tensor of a general (α, β)-metric.
Let

$$\bar{G}^i = G^i_\alpha + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.9}$$

Then (2.4) becomes

$$G^i = \bar{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i and $\bar{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.
Let

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.10}$$

Then $\bar{G}^i = G^i_\alpha + T^i$. Thus

$$\begin{aligned} D^i_{jkl} &= \bar{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right), \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{2.11}$$

To compute (2.11) explicitly, we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k,\ s_{y^k} = \alpha^{-2}(b_k\alpha - sy_k),$$

where $y_i = a_{il}y^l$. Hereafter, α_{y^k} means $\frac{\partial\alpha}{\partial y^k}$. Then

$$[\alpha Q s^m_0]_{y^m} = \alpha^{-1}y_m Q s^m_0 + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s^m_0 = Q's_0$$

and

$$[\Psi(-2Q\alpha s_0+r_{00})b^m]_{y^m} = \Psi'\alpha^{-1}(b^2-s^2)[r_{00}-2Q\alpha s_0]+2\Psi[r_0-Q'(b^2-s^2)s_0-Qss_0]$$

, where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.10), we have

$$T^m_{y^m} = Q's_0 + \Psi'\alpha^{-1}(b^2-s^2)[r_{00}-2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2-s^2)s_0 - Qss_0]. \tag{2.12}$$

Let L and $\bar{L}$ be two (α,β)-metrics, we assume that they have the same Douglas tensor, i.e. $D^i_{jkl} = \bar{D}^i_{jkl}$.
From (2.8) and (2.11), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i\right) = 0.$$

Then there exists a class of scalar function $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i, \tag{2.13}$$

where $H^i_{00} = H^i_{jk}(x)y^j y^k$, T^i and $T^m_{y^m}$ are given by (2.10) and (2.12) respectively.

3. Projective equivalence between Special (α,β)-metric and Kropina metric

In this section, we find the projective relation between special (α,β)-metric $L = \alpha + 2\beta + \frac{\beta^2}{\alpha}$ and Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a same underlying manifold M of dimension $n > 2$.

For (α,β)-metric $L=\alpha+2\beta+\frac{\beta^2}{\alpha}$, one can prove by (2.3) that L is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha<1$ for any $x\in M$. The geodesic coefficients are given by (2.4) with

$$\begin{aligned}\theta &= \frac{1-3s^2-2s^3}{(1+2s+s^2)(1+2b^2-3s^2)},\\ Q &= \frac{2+2s}{1-s^2},\\ \Psi &= \frac{1}{1+2b^2-3s^2}.\end{aligned} \tag{3.1}$$

For Kropina metric $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$, the geodesic coefficients are given by (2.4) with

$$\bar{Q}=-\frac{1}{2s},\qquad \bar{\theta}=-\frac{s}{\bar{b}^2},\qquad \bar{\Psi}=\frac{1}{2\bar{b}^2}. \tag{3.2}$$

In this paper, we assume that $\lambda=\frac{1}{n+1}$. Since the Douglas tensor is projective invariant, we have

Theorem 3.1. *Let $L=\alpha+2\beta+\frac{\beta^2}{\alpha}$ be an (α,β)-metric and $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be an Kropina metric on an n-dimensional manifold $M(n>2)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Then L and $\bar{L}$ have the same Douglas tensors if and only if they are all Douglas metrics.*

Proof: First we prove the sufficient condition.

Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i{}_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl}=0$ and $\bar{D}^i{}_{jkl}=0$, that is both F and $\bar{F}$ have same Douglas tensor, then (2.7) holds.
Plugging (3.1) and (3.2) into (2.13), we have

$$\begin{aligned}H^i_{00} &= \frac{A^i\alpha^9+B^i\alpha^8+C^i\alpha^7+D^i\alpha^6+E^i\alpha^5+F^i\alpha^4+G^i\alpha^3+H^i\alpha^2+I^i}{J\alpha^8+k\alpha^6+L\alpha^4+M\alpha^2+N}\\ &+ \frac{\bar{A}^i\bar{\alpha}^2+\bar{B}^i}{2\bar{b}^2\bar{\beta}},\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
A^i &= 2(1+2b^2)\{(1+2b^2)s^i_0 - 2b^i s_0\},\\
B^i &= (1+2b^2)\{2(1+2b^2)\beta s^i_0 - 4\beta s_0 b^i + r_{00}b^i - 2\lambda y^i(r_0+s_0)\},\\
C^i &= -2\beta[\beta(1+2b^2)\{6+(1+2b^2)s^i_0 - 2\beta(4+2b^2)s_0 b^i - 12b^2\lambda s_0 y^i\}],\\
D^i &= \beta[-\beta^2\{(2+4b^2)(7+2b^2)s^i_0 - 8(2+b^2)s_0 b^i\} + \beta\{(5+4b^2)r_{00}b^i\\
&- 2\lambda y^i((5+4b^2)r_0 + (5+16b^2)s_0)\} - 6b^2 r_{00}\lambda y^i],\\
E^i &= 6\beta^3[\beta\{(1+4b^2)s^i_0 + 2s_0 b^i\} - 4\lambda s_0 y^i(1+b^2)],\\
F^i &= \beta^3[6\beta^2\{(5+4b^2)s^i_0 + 2s_0 b^i\} + 12b^2 r_{00}\lambda y^i + \beta\{(7+2b^2)r_{00}b^i\\
&- 2\lambda y^i((7+2b^2)r_0 + (19+20b^2)s_0)\}],\\
G^i &= -6\beta^5[3\{\beta s^i_0 + 2s_0\lambda y^i\} - 10\lambda s_0 y^i],\\
H^i &= -3\beta^5[6\beta^2 s^i_0 + (4+2b^2)r_{00}\lambda y^i + \beta\{r_{00}b^i - 2\lambda y^i(r_0+5s_0)\}],\\
I^i &= 6\beta^7 r_{00}\lambda y^i,\\
J &= (1+2b^2)^2,\\
K &= -4\beta^2(1+2b^2)(2+b^2),\\
L &= \beta^4[(1+2b^2)(13+2b^2)+9],\\
M &= -12\beta^6(b^2+2),\\
N &= 9\beta^8,\\
\bar{A}^i &= \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0,\\
\bar{B}^i &= \bar{\beta}[2\lambda y^i(\bar{r}_0+\bar{s}_0) - \bar{b}^i\bar{r}_{00}].
\end{aligned}
$$

Further (3.3) is equivalent to

$$
\begin{aligned}
&(A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\\
&\times(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N). \quad (3.4)
\end{aligned}
$$

Replacing (y^i) by $(-y^i)$ in (3.4), we get

$$
\begin{aligned}
&(-A^i\alpha^9 + B^i\alpha^8 - C^i\alpha^7 + D^i\alpha^6 - E^i\alpha^5 + F^i\alpha^4 - G^i\alpha^3 + H^i\alpha^2 + I^i)(-2\bar{b}^2\bar{\beta}) - (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\\
&\times(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N) = -H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N). \quad (3.5)
\end{aligned}
$$

Adding (3.4) and (3.5) we obtain

$$
\begin{aligned}
&(A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3)(\bar{b}^2\bar{\beta}) = 0\\
&A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3 = 0.
\end{aligned}
$$

Therefore we conclude that (3.3) is equivalent to

$$
H^i_{00} = \frac{B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i}{J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} \quad (3.6)
$$

and (3.6) is equivalent to

$$
\begin{aligned}
&(B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N)\\
&= H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N). \quad (3.7)
\end{aligned}
$$

In the above equation (3.7), we can see that $\bar{A}^i\bar{\alpha}^2(J\alpha^8 + k\alpha^6 + L\alpha^4 + M\alpha^2 + N)$ can be divided by $\bar{\beta}$. Since $\beta = \mu\bar{\beta}$, then $\bar{A}^i\bar{\alpha}^2J\alpha^8$ can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to α and $\bar{\alpha}$. Therefore $\bar{A}^i = \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence there is a scalar function $\psi^i(x)$ such that

$$\bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0 = \bar{\beta}\psi^i. \tag{3.8}$$

Transvecting (3.8) by $\bar{y}_i = \bar{a}_{ij}y^j$, we get $\psi^i(x) = -\bar{s}^i$. Thus we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i). \tag{3.9}$$

Thus by lemma (2.2), $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metrics. i.e., Both $L = \alpha + 2\beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

If $n = 2$, $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric by lemma (2.2). Thus L and $\bar{L}$ have the same Douglas tensors means that they are Douglas metrics. Thus $L = \alpha+2\beta+\frac{\beta^2}{\alpha}$ be an special (α,β)-metric and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 2)$, where α and $\bar{\alpha}$ are Riemannian metric, β and $\bar{\beta}$ are two non zero collinear 1-forms. Then L and $\bar{L}$ have same Douglas tensors if and only if they are Douglas metrics. This completes the proof of theorem (3.1).

4. Proof. of Theorem (1.2)

First we prove the necessary condition:
Since Douglas tensor is an invariant under projective changes between two Finsler metrics, If L is projectively related to $\bar{L}$, then they have the same Douglas tensor. According to theorem (3.1), we obtain that both L and $\bar{L}$ are Douglas metrics. By [3], It is well know that Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ with $b^2 \neq 0$ is a Douglas metric if and only if $s_{ik} = \frac{1}{b^2}(b_is_k - b_ks_i)$ and also it has it has been proved that by [5], we know that (α,β)-metric, $L = \alpha + 2\beta + \frac{\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = 2\tau[(1 + 2b^2)a_{ij} - 3b_ib_j], \tag{4.1}$$

where $\tau = \tau(x)$ is a scalar function on M. In this case, β is closed.
Plugging (4.1) and (3.1) into (2.4), we have

$$G^i = G^i_\alpha + \left(\frac{\alpha^3 - 3\alpha\beta^2 - 2\beta^3}{\alpha^2 + \alpha\beta + \beta^2}\right) 2\tau y^i + 2\tau\alpha^2 b^i. \tag{4.2}$$

Again plugging (4.2) and (3.2) into (2.4), we have

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}\left[-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0y^i - \bar{r}_{00}\bar{b}^i) + 2\left(\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\right)\right]. \tag{4.3}$$

Since L is Projectively equivalent to $\bar{L}$, then there exist a scalar function $P = P(x,y)$ on $TM\backslash\{0\}$ such that

$$G^i = \bar{G}^i + Py^i. \tag{4.4}$$

By (4.2), (4.3) and (4.4), we have

$$\left[P-\left(\frac{\alpha^3-3\alpha\beta^2-2\beta^3}{\alpha^2+2\alpha\beta+\beta^2}\right)2\tau-\frac{1}{\bar{b}^2}\left(\bar{s}_0+\frac{\bar{r}_{00}\bar{\beta}}{\alpha^2}\right)\right]y^i=G^i_\alpha-\bar{G}^i_{\bar{\alpha}}+2\alpha^2\tau b^i-\frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i). \tag{4.5}$$

Note that RHS of above equation is in quadratic form.
Then there must be a one form $\theta=\theta_i y^i$ on M, such that

$$P-\left(\frac{\alpha^3-3\alpha\beta^2-2\beta^3}{\alpha^2+2\alpha\beta+\beta^2}\right)2\tau-\frac{1}{\bar{b}^2}\left(\bar{s}_0+\frac{\bar{r}_{00}\bar{\beta}}{\alpha^2}\right)=\theta.$$

Thus (4.5) becomes

$$G^i_\alpha+2\alpha^2\tau b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i)+\theta y^i. \tag{4.6}$$

This completes the proof of necessity.
Conversely from (4.2),(4.3) and (1.5) we have

$$G^i=\bar{G}^i+\left[\theta+\left(\frac{\alpha^3-3\alpha\beta^2-2\beta^3}{\alpha^2+2\alpha\beta+\beta^2}\right)2\tau-\frac{1}{\bar{b}^2}\left(\bar{s}_0+\frac{\bar{r}_{00}\bar{\beta}}{\alpha^2}\right)\right]y^i. \tag{4.7}$$

Thus L is projectively equivalent to $\bar{L}$. From the theorem (1.2), immediately we get the following corollary

Corollary 4.1. *: Let $L=\alpha+2\beta+\frac{\beta^2}{\alpha}$ ba special (α,β)-metric and $\bar{L}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric be two (α,β)-metrics on a n-dimensional manifold M with dimension $n>2$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero collinear 1-forms. Then L is projectively related to $\bar{L}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relations*

$$G^i+2\alpha^2\tau b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}[\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i]+\theta y^i,$$
$$s_{ij}=0,$$
$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i),$$
$$b_{i|j}=2\tau\{(1+2b^2)a_{ij}+3b_ib_j\}.$$

where $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α.

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