

THE STOLARSKY TYPE FUNCTIONS AND THEIR MONOTONICITIES

V. Lokesha^{*†}, Zhi-Gang Wang[‡], Zhi-Hua Zhang[§] and S. Padmanabhan[¶]

Received 04:07:2008 : Accepted 23:03:2009

Abstract

In this paper, we give the definition of a Stolarsky type function, and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in n variables.

Keywords: Two-parameter, Monotonicity, Mean, Vandermonde determinant.

2000 AMS Classification: 26D15.

1. Introduction

The so-called Stolarsky means $S(a, b; \alpha)$ were defined first by Stolarsky in [9] as follows:

$$(1.1) \quad S(a, b; \alpha) = \left[\frac{a^{\alpha+1} - b^{\alpha+1}}{(\alpha+1)(a-b)} \right]^{1/\alpha}, \quad \alpha(\alpha+1)(a-b) \neq 0;$$

$$(1.2) \quad S(a, b; -1) = \frac{a-b}{\ln a - \ln b}, \quad \alpha(a-b) \neq 0, \alpha = -1;$$

$$(1.3) \quad S(a, b; 0) = \exp \left(-1 + \frac{a \ln a - b \ln b}{a-b} \right), \quad (\alpha+1)(a-b) \neq 0, \alpha = 0;$$

$$(1.4) \quad S(a, a; \alpha) = a, \quad a = b.$$

The monotonicity of $S(a, b; \alpha)$ has been discussed by Leach and Sholander [3, 4], and by Qi [7, 8] also using different ideas and simpler methods.

^{*}Department of Mathematics, Acharya institute of Technology, Soldevahnalli, Hesaragatta Road, Karanataka Bangalore-90, India. E-mail lokiv@yahoo.com

[†]Corresponding author

[‡]School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha 410076, Hunan, People's Republic of China. E-mail: zhigwang@163.com

[§]Department of Mathematics, Zixing Educational Research Section, Chenzhou 423400, Hunan, People's Republic of China. E-mail: zxzh1234@163.com

[¶]Department of Mathematics, R.N.S. Institute of Technology, Karanataka, Bangalore, India. E-mail: padmanabhan_apsce@rediffmail.com

In [7], Qi studied the following generalized weighted Stolarsky type mean values $E_{f,p}(a, b; \alpha)$ with parameter α , and proved that $E_{f,p}(x, y; \alpha)$ is an increasing function in α :

$$(1.5) \quad E_{f,p}(a, b; \alpha) = \left(\frac{\int_a^b p(u) f^\alpha(u) du}{\int_a^b p(u) du} \right)^{\frac{1}{\alpha}}, \quad (\alpha - \beta)(a - b) \neq 0;$$

$$(1.6) \quad E_{f,p}(a, b; 0) = \exp \left(\frac{\int_a^b p(u) \ln f(u) du}{\int_a^b p(u) du} \right), \quad \alpha = 0, a - b \neq 0;$$

$$(1.7) \quad E_{f,p}(a, a; \alpha) = f(a), \quad \alpha = \beta, a = b;$$

where $a, b, \alpha, \beta \in \mathbb{R}$, $p \geq 0$, and $f > 0$ is an integrable function on the interval $[a, b] \subset \mathbb{R}$.

We know by the definition of the power mean that

$$(1.8) \quad M(x; \alpha) = \left(\frac{\sum_{k=1}^n x_k^\alpha}{n} \right)^{\frac{1}{\alpha}}, \quad \alpha \neq 0;$$

$$(1.9) \quad M(x; 0) = \exp \left(\frac{\sum_{k=1}^n \ln x_k}{n} \right), \quad \alpha = 0;$$

where $a_k \in \mathbb{R}_+$, and $\alpha \in \mathbb{R}$.

We note that for each of these two means the one-parameter means are of the type $(F(\alpha)/F(0))^{1/\alpha}$ if $\alpha \neq 0$, and $\exp(F'(\alpha)/F(\alpha))$ if $\alpha = 0$, where $F(\alpha)$ is a certain univariate function involving an α -order power.

In this paper, we define a Stolarsky type function and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in n variables.

2. Main results

Throughout the paper we assume \mathbb{R} to be the set of real numbers, \mathbb{R}_+ the set of strictly positive real numbers, \mathbb{R}^n the n -dimensional Euclidean Space,

$$\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) : x_i > 0, i = 1, 2, \dots, n\},$$

and

$$\begin{aligned} \alpha x &= (\alpha x_1, \alpha x_2, \dots, \alpha x_n), & e^x &= (e^{x_1}, e^{x_2}, \dots, e^{x_n}), \\ x^\alpha &= (x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha), & \ln x &= (\ln x_1, \ln x_2, \dots, \ln x_n), \end{aligned}$$

where $\alpha \in \mathbb{R}$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}_+^n$.

2.1. Definition. Let $\alpha, \beta \in \mathbb{R}$, and f be continuous involving an $(\alpha\beta)$ -order power function on $I \subseteq \mathbb{R}_+^n$. If $F(\alpha) = f(x; \alpha\beta)$, $\beta \neq 0$, and f is a differentiable function with respect to $\alpha \in \mathbb{R}$, then the Stolarsky type function $S_f(x; \alpha, \beta)$ is defined as follows,

$$(2.1) \quad S_f(x; \alpha) = \left(\frac{f(x; \alpha\beta)}{f(x; 0)} \right)^{\frac{1}{\alpha}}, \quad (\alpha \neq 0),$$

$$(2.2) \quad S_f(x; 0) = \lim_{\alpha \rightarrow 0} \exp \left(\frac{f'_\alpha(x; \alpha\beta)}{f(x; \alpha\beta)} \right), \quad (\alpha = 0),$$

where f'_α is the partial derivative with respect to α of $f(x; \alpha\beta)$.

2.2. Remark. For convenience, we write

$$(2.3) \quad S_f(x; \alpha) = S_f(x) = S_f(\alpha) = S_f,$$

shifting notation to suit the context.

2.3. Theorem. Let $\alpha, \beta \in \mathbb{R}, \beta \neq 0$, and f be continuous involving an $(\alpha\beta)$ -order power function on $I \subseteq \mathbb{R}_+^n$. If

$$(2.4) \quad f(x; \alpha\beta) f''_{\alpha\alpha}(x; \alpha\beta) > [f'_\alpha(x; \alpha\beta)]^2,$$

then $S_f(x; \alpha)$ is a monotonic increasing function in α , and monotonic decreasing if the inequality (2.4) is reversed.

Proof. Suppose the inequality (2.4) holds. Setting $T(\alpha) = \ln |f(x; \alpha\beta)|$, then $T'(\alpha) = f'_\alpha(x; \alpha\beta)/f(x; \alpha\beta)$, and

$$T''(\alpha) = \frac{f(x; \alpha\beta) f''_{\alpha\alpha}(x; \alpha\beta) - [f'_\alpha(x; \alpha\beta)]^2}{[f(x; \alpha\beta)]^2} > 0.$$

When $\alpha = 0$, $\ln S_f = f'_\alpha(x; \alpha\beta)/f(x; \alpha\beta) = T'(\alpha)$, and $\partial \ln S_f / \partial \alpha = T''(\alpha) > 0$, which implies that $S_f(x; \alpha)$ is a monotonic increasing function in α .

When $\alpha \neq 0$, using the mean value theorem, we find

$$\frac{\partial \ln S_f}{\partial \alpha} = \frac{T'(\alpha)}{\alpha} - \frac{T(\alpha)}{\alpha^2} = \frac{T'(\alpha) - T(\alpha)/\alpha}{\alpha} = \frac{T'(\alpha) - T'(\zeta)}{\alpha} = \frac{\alpha - \eta}{\alpha} T''(\eta) > 0,$$

where ζ is between 0 and α , and η is between α and ζ . That is to say, $S_f(x; \alpha)$ is a monotonic increasing function in α . Theorem 2.3 is thus proved. \square

3. The generalized weighted Stolarsky type functional mean

3.1. Theorem. The generalized weighted Stolarsky type functional mean values $S_{f,p}(x; \alpha)$ are monotonic increasing functions with α in \mathbb{R} , where

$$(3.1) \quad S_{f,p}(x; \alpha) = \left(\frac{\int_E p(t) f^\alpha(A(x; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \neq 0,$$

$$(3.2) \quad S_{f,p}(x; 0) = \exp \left(\frac{\int_E p(t) \ln f(A(x; t)) dt}{\int_E p(t) dt} \right), \quad \alpha = 0,$$

and $E = \{(t_1, t_2, \dots, t_n) \mid \sum_{i=1}^n t_i \leq 1, t_i \geq 0, i = 1, 2, \dots, n\}$, $t_0 = 1 - \sum_{i=1}^n t_i$, $A(x; t) = x_0 + \sum_{i=1}^n (x_i - x_0) t_i = \sum_{i=0}^n x_i t_i$, $x_i \in I \subseteq \mathbb{R}_+$, and $p \geq 0, f > 0$ integrable functions respectively on E and I .

Proof. By taking $T(x; \alpha) = \int_E p(t) f^\alpha(A(x; t)) dt$, and using Cauchy's integral inequality, we have

$$\begin{aligned} & T(x; \alpha\beta) T''_{\alpha\alpha}(x; \alpha\beta) - [T'_\alpha(x; \alpha\beta)]^2 \\ &= \int_E p(t) f^\alpha(A(x; t)) dt \cdot \int_E p(t) f^\alpha(A(x; t)) \ln^2 f(A(x; t)) dt \\ &\quad - \left(\int_E p(t) f^\alpha(A(x; t)) \ln f(A(x; t)) dt \right)^2 > 0, \end{aligned}$$

which implies Theorem 3.1 from Theorem 2.3. \square

3.2. Corollary. The generalized weighted Stolarsky type functional mean values $E_{f,p}(a, b; \alpha)$ are monotonic increasing functions with α in \mathbb{R} , where $E_{f,p}(a, b; \alpha)$ is given by (1.5)–(1.7).

Proof. Setting $u = x_0 + (x_1 - x_0)t_1$, then $du = (x_1 - x_0)dt_1$. Setting $a = x_0$ and $b = x_1$, from Theorem 3.1, we immediately obtain Corollary 3.2. The proof is completed. \square

4. The generalized weighted Stolarsky type functional mean with two parameters

4.1. Definition. Let $\alpha, \beta \in R$, E , t_0 and p, f be defined as in Theorem 3.1. If

$$M_\beta(x; t) = \left(x_0^\beta + \sum_{i=1}^n (x_i^\beta - x_0^\beta) t_i \right)^{1/\beta} = \left(\sum_{i=0}^n x_i^\beta t_i \right)^{1/\beta},$$

and $M_0(x; t) = G(x; t) = \prod_{i=0}^n x_i^{t_i}$, then the first generalized weighted Stolarsky type functional mean values, $S_{f,p}^{[1]}(x; \alpha, \beta)$, with two parameters α and β are as follows

$$(4.1) \quad S_{f,p}^{[1]}(x; \alpha, \beta) = \left(\frac{\int_E p(t) f^\alpha(M_\beta(x; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha\beta \neq 0;$$

$$(4.2) \quad S_{f,p}^{[1]}(x; 0, \beta) = \exp \left(\frac{\int_E p(t) \ln f(M_\beta(x; t)) dt}{\int_E p(t) dt} \right), \quad \alpha = 0, \beta \neq 0;$$

$$(4.3) \quad S_{f,p}^{[1]}(x; \alpha, 0) = \left(\frac{\int_E p(t) f^\alpha(G(x; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \neq 0, \beta = 0;$$

$$(4.4) \quad S_{f,p}^{[1]}(x; 0, 0) = \exp \left(\frac{\int_E p(t) \ln f(G(x; t)) dt}{\int_E p(t) dt} \right), \quad \alpha = \beta = 0.$$

In a manner similar to Section 3, from Definition 4.1 we obtain the following theorem.

4.2. Theorem. *The first generalized weighted Stolarsky type functional mean values $S_{f,p}^{[1]}(x; \alpha, \beta)$ are monotonic increasing functions in $\alpha \in R$.*

4.3. Theorem. *The first generalized weighted Stolarsky type functional mean values $S_{f,p}^{[1]}(x; \alpha, \beta)$ are monotonic increasing functions with $\beta \in R$ if f is a monotonic increasing function.*

Proof. This follows from the weighted power mean inequality, Definition 4.1 and the fact that f is a monotonic increasing function. \square

4.4. Remark. We have $S_{f,p}^{[1]}(x; \alpha, 1) = S_{f,p}(x; \alpha)$.

4.5. Definition. Let $\alpha, \beta \in R$, E , t_0 and p, f be defined as in Theorem 3.1. If

$$M_\beta(x^\alpha; t) = \left[x_0^{\alpha\beta} + \sum_{i=1}^n (x_i^{\alpha\beta} - x_0^{\alpha\beta}) t_i \right]^{1/\beta} = \left(\sum_{i=0}^n x_i^{\alpha\beta} t_i \right)^{1/\beta},$$

$M_0(x^\alpha; t) = G(x^\alpha; t) = \prod_{i=0}^n x_i^{\alpha t_i}$, and $f'(1)$ exists, then the second generalized weighted Stolarsky type functional mean values $S_{f,p}^{[2]}(x; \alpha, \beta)$ with two parameters α and β are as follows

$$(4.5) \quad S_{f,p}^{[2]}(x; \alpha, \beta) = \left(\frac{\int_E p(t) f(M_\beta(x^\alpha; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha\beta \neq 0;$$

$$(4.6) \quad S_{f,p}^{[2]}(x; 0, \beta) = \exp \left(\frac{\int_E p(t) f'(1) \left(\sum_{k=1}^n t_k \ln x_k \right) dt}{\int_E p(t) dt} \right), \quad \alpha = 0, \beta \in \mathbb{R};$$

$$(4.7) \quad S_{f,p}^{[2]}(x; \alpha, 0) = \left(\frac{\int_E p(t) f(G(x^\alpha; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \neq 0, \beta = 0.$$

4.6. Theorem. *The second generalized weighted Stolarsky type functional mean values $S_{f,p}^{[2]}(x; \alpha, \beta)$ are monotonic increasing functions with α in R if $f' > 0$, $ff'' > (f')^2$ and $\beta > 0$.*

Proof. By taking $T(x; \alpha\beta) = \int_E p(t)f(M_\beta(x^\alpha; t))dt$, if $\beta \neq 0$, then

$$(4.8) \quad T'_\alpha(x; \alpha\beta) = \int_E p(t)f'(M_\beta(x^\alpha; t)) \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-1} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right) dt,$$

$$(4.9) \quad T''_\alpha(x; \alpha\beta) = \int_E p(t)f''(M_\beta(x^\alpha; t)) \left[\left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-1} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right) \right]^2 dt \\ + \int_E p(t)f'(M_\beta(x^\alpha; t)) \left[(1-\beta) \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-2} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right)^2 \right. \\ \left. + \beta \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-1} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln^2 x_k \right) \right] dt.$$

Using Cauchy's integral inequality, from (4.8)-(4.9), and $f' > 0, ff'' > (f')^2, \beta > 0$, yields

$$T(x; \alpha\beta)T''_{\alpha\alpha}(x; \alpha\beta) - [T'_\alpha(x; \alpha\beta)]^2 \\ = \int_E p(t)f(M_\beta(x^\alpha; t))dt \cdot \int_E p(t)f''(M_\beta(x^\alpha; t)) \\ \cdot \left[\left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-1} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right) \right]^2 dt \\ - \left[\int_E p(t)f'(M_\beta(x^\alpha; t)) \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-1} \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right) dt \right]^2 \\ + \int_E p(t)f(M_\beta(x^\alpha; t))dt \cdot \int_E p(t)f''(M_\beta(x^\alpha; t)) \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right)^{1/\beta-2} \\ \cdot \left\{ \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right)^2 + \beta \left[\left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \right) \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln^2 x_k \right) \right. \right. \\ \left. \left. - \left(\sum_{k=0}^n x_k^{\alpha\beta} t_k \ln x_k \right)^2 \right] \right\} dt > 0$$

which implies Theorem 4.6 from Theorem 2.3. If $\beta = 0$ we can obtain Theorem 4.6 similarly. \square

5. Some mean values in n variables

5.1. Notation and lemmas. Throughout this section we assume $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}_+^{n+1}$, and that φ is a function in \mathbb{R} . Put

$$(5.1) \quad V(x; \varphi) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & \varphi(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & \varphi(x_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & \varphi(x_n) \end{vmatrix}.$$

Assuming $\varphi(t) = t^{n+r} \ln^k t$, then

$$(5.2) \quad V(x; r, k) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^{n+r} \ln^k x_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^{n+r} \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^{n+r} \ln^k x_n \end{vmatrix}.$$

Note the case $r = 0$ and $k = 0$ is just the determinant of Van der Monde's matrix of the n -th order:

$$(5.3) \quad V(x; 0, 0) = \prod_{0 \leq i < j \leq n} (x_j - x_i).$$

Write $\ln x = (\ln x_0, \ln x_1, \dots, \ln x_n)$, then

$$(5.4) \quad V(\ln x; r, k) = \begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r \ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r \ln^k x_n \end{vmatrix}.$$

Also, let $0 \leq i \leq n$, and set

$$(5.5) \quad V_{[i]}(x; \varphi) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & \varphi(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & \varphi(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_i & x_i^2 & \cdots & x_i^{n-1} & \varphi(x_i) \\ 0 & 1 & 2x_i & \cdots & (n-1)x_i^{n-2} & \varphi'(x_i) \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-1} & \varphi(x_{i+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & \varphi(x_n) \end{vmatrix}$$

and for $\varphi(t) = t^{n+r+1}$ in (5.5), we have

$$(5.6) \quad V_{[i]}(x; r) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n & x_0^{n+r+1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^n & x_1^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_i & x_i^2 & \cdots & x_i^n & x_i^{n+r+1} \\ 0 & 1 & 2x_i & \cdots & nx_i^{n-1} & (n+r+1)x_i^{n+r} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^n & x_{i+1}^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n & x_n^{n+r+1} \end{vmatrix}, \quad (i \leq i \leq n),$$

and

$$(5.7) \quad V_{[i]}(x; 0) = (-1)^{i+1} V(x; 0, 0) \prod_{j=0, j \neq i}^n (x_j - x_i) = (-1)^{i+1} V^2(x; 0, 0) / V_i(x),$$

where

$$(5.8) \quad V_i(x) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{i-1} & x_{i-1}^2 & \cdots & x_{i-1}^{n-1} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}, \quad (0 \leq i \leq n).$$

5.1. Lemma. (see [12, 13, 14]) *If $n \in \mathbb{N}$, and φ is a n -order differentiable function on an interval $I \subset \mathbb{R}_+$, then,*

$$(5.9) \quad V(x; \varphi) = V(x; 0, 0) \int_E \varphi^{(n)}(A(x; t)) dt,$$

$$(5.10) \quad \sum_{i=0}^n (-1)^{i+1} \lambda_i V_{[i]}(x; \varphi) V_i(x) = V^2(x; 0, 0) \int_E A(\lambda; t) \varphi^{(n)}(A(x; t)) dx,$$

where $dt = dt_1 dt_2 \cdots dt_n$, and $E, A(x; t)$ are as in Theorem 3.1.

5.2. Lemma. (see [10]) *Let r be an integer, then*

$$(5.11) \quad V(a; r, 0) = V(a; 0, 0) \cdot \sum_{i_0+i_1+\dots+i_n=r}^{i_0, i_2, \dots, i_n \geq 0} \prod_{k=0}^n a_k^{i_k}, \quad r > 0;$$

$$(5.12) \quad V(a; r, 0) = 0, \quad r = 0, 1, \dots, -(n-1);$$

$$(5.13) \quad V(a; r, 0) = (-1)^n V(a; 0, 0) \cdot \sum_{i_0+i_1+\dots+i_n=-r}^{i_0, i_1, \dots, i_n \geq 1} \prod_{k=0}^n a_k^{-i_k}, \quad r < -n.$$

5.2. The Stolarsky type mean with one parameter in n variables.

5.3. Definition. (see [11]) *The Stolarsky type generalized mean values $S_\alpha(x)$ with parameter α in n variables are*

$$(5.14) \quad S_\alpha(x) = \left[n! \int_E \varphi_1^{(n)}(\alpha, A(x; t)) dt \right]^{1/\alpha}, \quad \alpha \neq 0,$$

$$(5.15) \quad S_0(x) = \exp \left(n! \int_E \varphi_2^{(n)}(0, A(x; t)) dt \right), \quad \alpha = 0,$$

where $\varphi_1^{(n)}(\alpha, t) = t^\alpha$ and $\varphi_2^{(n)}(\alpha, t) = t^\alpha \ln t$.

5.4. Theorem. *If the generalized mean values $S_\alpha(x)$ with two parameters α and β , in n variables are as given by Definition 5.3, then*

$$(5.16) \quad S_\alpha(x) = \left[\frac{n!}{\prod_{k=1}^n (k + \alpha)} \cdot \frac{V(x; \alpha, 0)}{V(x; 0, 0)} \right]^{\frac{1}{\alpha}}, \quad \alpha \neq 0, -1, -2, \dots, -n;$$

$$(5.17) \quad S_0(x) = \exp \left(\frac{V(x; 0, 1)}{V(x; 0, 0)} - \sum_{k=1}^n \frac{1}{k} \right), \quad \alpha = 0;$$

$$(5.18) \quad S_\alpha(x) = \left[\frac{n! \cdot V(x; \alpha, 1)}{(-1)^{\alpha+1} (-\alpha - 1)! \cdot (n + \alpha)! \cdot V(x; 0, 0)} \right]^{\frac{1}{\alpha}}, \quad \alpha = -1, \dots, -n;$$

where $S_\alpha(x)$ are monotonic increasing functions with α in R .

Proof. Consider the following two functions:

$$(5.19) \quad \varphi_1(\alpha, t) = \prod_{k=1}^n (k + \alpha)^{-1} t^{n+\alpha},$$

if $\alpha \neq 0, -1, -2, \dots, -n$; and

$$(5.20) \quad \varphi_2(0, t) = (n!)^{-1} t^n \left(\ln t - \sum_{k=1}^n \frac{1}{k} \right),$$

if $\alpha = 0$; and

$$(5.21) \quad \varphi_1(\alpha, t) = [(-1)^{\alpha+1} (-\alpha - 1)! (n + \alpha)!]^{-1} t^{n+\alpha} \ln t,$$

if $\alpha = -1, -2, \dots, -n$. Then $\varphi_1^{(n)}(\alpha, t) = t^\alpha$ and $\varphi_2^{(n)}(0, t) = \ln t$.

According to Lemma 5.1 and (5.19)–(5.21), we know that the expressions (5.16)–(5.18) hold true.

Let $f^\alpha(A(x; t)) = \varphi_1^{(n)}(\alpha, A(x; t))$. Then $\ln f(A(x; t)) = \varphi_2^{(n)}(0; A(x; t))$. Taking $p(x) \equiv 1$, we find from Theorem 3.1 that the $S_\alpha(x)$ are monotonic increasing functions with α in R . The proof of Theorem 5.4 is completed. \square

5.5. Remark. (see [10]–[17]) $S_0(x)$ is the so-called identric mean in n variables, and $S_{-1}(x)$ the first logarithmic mean $L(x)$. It is noted that $S_0(x) := I(x)$, and

$$(5.22) \quad L(x) := S_{-1}(x) = \frac{V(x; 0, 0)}{nV(x; -1, 1)}.$$

5.6. Remark. (see [1]) If α is a nonnegative integer, from Lemma 5.2, $[S_\alpha(x)]^\alpha$ is the r -th generalized elementary symmetric mean in n variables, i.e.

$$(5.23) \quad \sum_n^{[\alpha]}(x) := [S_\alpha(x)]^\alpha = \binom{n+\alpha}{\alpha}^{-1} \sum_{\substack{i_0+i_1+\dots+i_n=\alpha, \\ i_0, i_1, \dots, i_n \in \mathbb{N}_0}} \prod_{k=1}^n x_k^{i_k}.$$

5.3. The Stolarsky type mean with two parameters in n variables.

5.7. Definition. (see [12]) The Stolarsky type generalized mean values $S_{\alpha, \beta}(x)$ with two parameters α and β in n variables are

$$(5.24) \quad S_{\alpha, \beta}(x) = \left[n! \int_E \varphi_1^{(n)}(\alpha, M_\beta(x; t)) dt \right]^{1/\alpha}, \quad \alpha \neq 0, \beta \neq 0;$$

$$(5.25) \quad S_{0, \beta}(x) = \exp \left(n! \int_E \varphi_2^{(n)}(0, M_\beta(x; t)) dt \right), \quad \alpha = 0, \beta \neq 0;$$

$$(5.26) \quad S_{\alpha, 0}(x) = \left[n! \int_E \varphi_1^{(n)}(\alpha, G(x; t)) dt \right]^{1/\alpha}, \quad \alpha \neq 0, \beta = 0;$$

$$(5.27) \quad S_{0, 0}(x) = \left(\prod_{i=0}^n a_i \right)^{1/(n+1)}, \quad \alpha = 0, \beta = 0;$$

where $\varphi_1^{(n)}(\alpha, t) = t^\alpha$ and $\varphi_2^{(n)}(\alpha, t) = t^\alpha \ln t$.

5.8. Theorem. We have that $S_{\alpha, \beta}(x)$ are monotonic increasing functions with α in \mathbb{R} , and

$$(5.28) \quad S_{\alpha, \beta}(x) = \left[\frac{n! \cdot \beta^n}{\prod_{k=1}^n (k\beta + \alpha)} \cdot \frac{V(x^\beta; \alpha/\beta, 0)}{V(x^\beta; 0, 0)} \right]^{1/\alpha}, \quad \beta \neq 0, \alpha \neq -k\beta, 0 \leq k \leq n;$$

$$(5.29) \quad S_{\alpha, \beta}(x) = \left[(-1)^{k+1} k\beta \binom{n}{k} \frac{V(x^\beta; -k, 1)}{V(x^\beta; 0, 0)} \right]^{-1/(k\beta)}, \quad \beta \neq 0, \alpha = -k\beta, 1 \leq k \leq n;$$

$$(5.30) \quad S_{\alpha, 0}(x) = \left[\frac{n!}{\alpha^n} \cdot \frac{V(\ln x; \alpha, 0)}{V(\ln x; 0, n)} \right]^{1/\alpha}, \quad \beta = 0, \alpha \neq 0;$$

$$(5.31) \quad S_{0, \beta}(x) = \exp \left(\frac{V(x^\beta; 0, 1)}{V(x^\beta; 0, 0)} - \frac{1}{\beta} \sum_{k=1}^n \frac{1}{k} \right), \quad \alpha = 0, \beta \neq 0;$$

$$(5.32) \quad S_{0, 0}(x) = \left(\prod_{i=0}^n x_i \right)^{1/(n+1)}, \quad \alpha = \beta = 0.$$

5.9. Remark. Replacing α by $\alpha - \beta$, the generalized Stolarsky type mean $S_{\alpha-\beta,\beta}(x)$ is the Pečarić-Šimić mean in [6].

5.10. Remark. (see [15] and also [5, 16]) If $\alpha = 1$, then $S_{1,0}(x)$ is the second logarithmic mean in n variables:

$$(5.33) \quad l(x) := S_{1,0}(x) = \frac{n!V(\ln x; 1, 0)}{V(\ln x; 0, n)}.$$

5.11. Remark. (see [15] and also [5]) Change β to $1/\beta$, and set $\alpha = 1$. If β is a nonnegative integer, from Lemma 5.2 we see that $S_{1,1/\beta}(x)$ is the generalized Heron's mean in n variables:

$$(5.34) \quad H_\beta(x) := S_{1,1/\beta}(x) = \binom{n+\beta}{\beta}^{-1} \sum_{\substack{i_0+i_1+\dots+i_n=\beta, \\ i_0, i_1, \dots, i_n \in \mathbb{N}_0}} \prod_{k=1}^n x_k^{i_k/\beta},$$

5.4. The r -th weighted elementary symmetric mean in n variables.

5.12. Definition. (see [17]) Let x be a tuple of n non-negative real numbers and the weight λ a tuple of n positive real numbers, then

$$(5.35) \quad E_\alpha(x, \lambda) = \sum_{\substack{i_0+i_1+\dots+i_n=\alpha, \\ i_0, i_1, \dots, i_n \in \mathbb{N}_0}} \sum_{k=0}^n \lambda_k (1+i_k) x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$$

is called the α -th weighted elementary symmetric function of x for the positive weight λ , where the sum is over all $(n+\alpha+1)$ -tuples of non-negative integers such that $i_1 + i_2 + \cdots + i_n = \alpha$; In addition, $E_0(x, \lambda) = \sum_{i=1}^n \lambda_i$. The α -th weighted elementary symmetric mean of x for λ is defined by

$$(5.36) \quad \sum_\alpha(x, \lambda) = \frac{E_\alpha(x, \lambda)}{\binom{n+\alpha+1}{\alpha} \sum_{i=1}^n \lambda_i}.$$

5.13. Theorem. (see [17]) If $r \in \mathbb{N}$, then

$$(5.37) \quad \sum_\alpha(x, \lambda) = \frac{\int_E \left(\sum_{i=0}^n \lambda_i x_i \right) \left(\sum_{i=0}^n a_i x_i \right)^\alpha dx}{\int_E \left(\sum_{i=0}^n \lambda_i x_i \right) dx}$$

is a monotonic increasing function with α in \mathbb{N} .

Acknowledgement. We are thankful to the referee for some valuable suggestions.

References

- [1] Hardy, G. H., Littlewood, J. E. and Polya, G. *Inequalities* (2nd.ed., Cambridge University Press, Cambridge, 1959).
- [2] Kuang, J.-C. *Applied Inequalities* (Hunan Education Press, 2nd. Ed., 1993), 39–49. (in Chinese).
- [3] Leach, E. B. and Sholander, M. C. *Extended mean values*, Amer. Math. Monthly **85**, 84–90, 1978.
- [4] Leach, E. B. and Sholander, M. C. *Extended mean values*, J. Math. Anal. Appl. **92**, 207–223, 1983.
- [5] Loksha, V. and Zhang, Z.-H. *The weighted Heron dual mean in n variables*, Adv. Stud. Contemp. Math. (Kyungshang) **13** (2), 165–170, 2006.
- [6] Pečarić, J. and Šimić, V. *Stolarsky-Tobey mean in n variables*, Math. Inequal. Appl. **2** (3), 328–341, 1999.
- [7] Qi, F. *Generalized weighted mean values with two parameters*, Proc. Royal Soc. London, Series, Math. Physical Engineering Sci. **454**, 2723–2732, 1998.

- [8] Qi, F. *On a two-parameter family of nonhomogeneous mean values*, Tamkang J. Math. **29**, 155–163, 1998.
- [9] Stolarsky, K. B. *Generalizations of the logarithmic mean*, Math. Mag. **48**, 87–92, 1975.
- [10] Xiao, Z.-G. and Zhang, Z.-H. *The Henleman mean of n positive numbers*, J. Yueyang Normal Univ. **14** (2), 1–5, 2001 (in Chinese).
- [11] Xiao, Z.-G. and Zhang, Z.-H. *The Stolarsky mean of n positive numbers*, J. Yueyang Normal Univ. **14** (4), 5–8, 2001 (in Chinese).
- [12] Xiao, Z.-G., Zhang, Z.-H., Lokesha, V. and Nagaraja, K. M. *A class of new three-parameter generalized weighted means*, Int. J. Appl. Math. Stat. **11** (7), 193–202, 2007.
- [13] Xiao, Z.-G., Zhang, Z.-H., Lokesha, V. and Tang, R. *The two-parameter mean of n variables*, Int. Rev. Pure Appl. Math. **1** (1), 93–111, 2005.
- [14] Xiao, Z.-G., Zhang, Z.-H. and Qi, F. *A type of mean values of several positive numbers with two parameters*, Nonlinear Funct. Anal. Appl. **12**, 687–702, 2007.
- [15] Zhang, Z.-H. *Three classes of new means in $n+1$ variables and their applications*, J. Hunan Ed. Inst. **15** (5), 130–136, 1997 (in Chinese).
- [16] Zhang, Z.-H., Lokesha, V. and Xiao, Z.-G. *The weighted Heron mean in n variables*, J. Anal. Comput. **1** (1), 57–68, 2005.
- [17] Zhang, Z.-H., Xiao, Z.-G. and Srivastava, H. M. *A general family of weighted elementary symmetric means*, Appl. Math. Lett. (2008), doi:10.1016/j.aml.2007.12.030