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A new refinement of the Janous–Gmeiner inequality for a triangle

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ABSTRACT

In this paper, the authors give a new refinement of the Janous–Gmeiner inequality for a triangle by making use of certain analytical techniques for systems of nonlinear algebraic equations. Some other closely-related geometric inequalities are also considered.

1. Introduction, preliminaries and the main result

For a given $\triangle ABC$, let a, b and c denote the side-lengths facing the angles A, B and C, respectively. Also let m_a, m_b and m_c denote the corresponding medians, $s = \frac{1}{2}(a+b+c)$ the semi-perimeter, R the circumradius and r the inradius of $\triangle ABC$. As long ago as 1986, Janous [1] posed the following conjecture involving a geometrical inequality:

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s}. \tag{1.1}$$

Later, in the year 1988, Gmeiner and Janous [2] proved the inequality (1.1) by using calculus. In 1989, Shan and Liu [3] also *independently* proved the inequality (1.1) by using calculus techniques. Moreover, Shan and Liu [3] pointed out that the following inequality does not hold true:

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geqq \frac{3\sqrt{3}}{s}. \tag{1.2}$$

Motivated by the work of Shan and Liu [3], An [4] considered the inequality (1.2) and proved the following inequality:

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geqq \frac{3\sqrt{3}}{s + \frac{1}{\sqrt{6}}(|a-b| + |b-c| + |c-a|)}. \tag{1.3}$$

Subsequently, Shi [5] refined the inequality (1.3) as follows:

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geqq \frac{3\sqrt{3}}{s + \frac{3\sqrt{3}-5}{10}(|a-b| + |b-c| + |c-a|)}, \tag{1.4}$$

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which also sharpened the inequality (1.1). Shi [6], on the other hand, obtained the following result:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \geqq \frac{3\sqrt{3}}{M_k(a,b,c)} \qquad \left(k \geqq \frac{\ln 9-\ln 4}{\ln 25-\ln 12}\right), \tag{1.5}$$

where, for convenience, $M_k(a, b, c)$ is given by

$$M_k(a,b,c) := \left(\frac{a^k+b^k+c^k}{3}\right)^{\frac{1}{k}} \qquad \left(k \geqq \frac{\ln 9-\ln 4}{\ln 25-\ln 12}\right). \tag{1.6}$$

In the same year 1996, Yang [7] improved the inequality (1.1) as follows:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \geqq \frac{5}{s}+(6\sqrt{3}-10)\frac{r}{Rs}. \tag{1.7}$$

Analytic as well as geometric inequalities are potentially useful in many different areas of the mathematical, physical and engineering sciences (see, for details, [8,9]; see also [10]). With this objective in view, we present a new refinement of the Janous–Gmeiner inequality (1.2) as asserted by the following theorem.

Theorem. *The best constant k for the following inequality:*

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \geqq \frac{3\sqrt{3}}{s+k(s-3\sqrt{3}r)} \tag{1.8}$$

is given by

$$k=\frac{3\sqrt{3}}{5}-1. \tag{1.9}$$

2. A set of lemmas

In order to prove our main result asserted by the Theorem in the preceding section, we require each of the following four lemmas.

Lemma 1. *The following implication holds true:*

$$r \leqq \frac{a\sqrt{s(s-a)}}{2s} \Longleftrightarrow -r \geqq -\frac{a\sqrt{s(s-a)}}{2s} \tag{2.1}$$

with equality if and only if $b = c$.

Proof. Making use of the familiar formula:

$$r=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \tag{2.2}$$

and the well-known AM–GM inequality, we easily obtain the inequality (2.1). Furthermore, it is not difficult to observe that the equality in (2.1) holds true if and only if $b = c$. □

Lemma 2 (*see [6]*). *If* $a \leqq b \leqq c$, *then*

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \geqq \frac{1}{\sqrt{s(s-a)}}+\frac{4}{\sqrt{2a^2+\frac{(b+c)^2}{4}}}, \tag{2.3}$$

where the equality holds true if and only if $b = c$.

Lemma 3 (*see [11] and [12]*). *Suppose that* $f(x)$ *is a polynomial with real coefficients given by*

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n. \tag{2.4}$$

If the number of the sign changes of the revised sign list of its discriminant sequence

$$\{D_1(f), D_2(f), \ldots, D_n(f)\} \tag{2.5}$$

is v, *then the number of the pairs of distinct conjugate imaginary roots of* $f(x)$ *equals* v. *Furthermore, if the number of non-vanishing members of the revised sign list is* ℓ, *then the number of the distinct real roots of* $f(x)$ *equals* $\ell - 2v$.

Lemma 4 (*see [12]*). *Let the polynomials $F(x)$ and $G(x)$ be given by*

$$F(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \tag{2.6}$$

and

$$G(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \tag{2.7}$$

respectively. If

$$a_0 \neq 0 \quad or \quad b_0 \neq 0, \tag{2.8}$$

then the polynomials $F(x)$ and $G(x)$ have common roots if and only if

$$R(F, G) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0, \tag{2.9}$$

where $R(F, G)$ is Sylvester's resultant of the polynomials $F(x)$ and $G(x)$.

3. Proof of the Theorem

For the symmetry of the inequality (1.8), there is no harm in assuming that $a \leqq b \leqq c$. Thus, by Lemmas 1 and 2, we only need to consider the best constant k for the following inequality:

$$\frac{1}{\sqrt{s(s-a)}} + \frac{4}{\sqrt{2a^2 + \frac{(b+c)^2}{4}}} \geqq \frac{3\sqrt{3}}{s + k\left[s - 3\sqrt{3}\left(\frac{a\sqrt{s(s-a)}}{2s}\right)\right]}. \tag{3.1}$$

Without loss of generality, we can set

$$\frac{b+c}{2} = 1 \quad \text{and} \quad a = x \quad (0 < x \leqq 1). \tag{3.2}$$

Then, clearly,

$$s = \frac{x+2}{2} \quad (0 < x \leqq 1)$$

and the inequality (3.1) is equivalent to

$$\frac{2}{\sqrt{4-x^2}} + \frac{4}{\sqrt{2x^2+1}} \geqq \frac{6\sqrt{3}}{x + 2 + k\left(x + 2 - \frac{3x\sqrt{3(4-x^2)}}{x+2}\right)}. \tag{3.3}$$

We consider the following two cases separately.

Case 1. When $x = 1$, the inequality (3.3) holds true for any $k \in \mathbb{R} := (-\infty, \infty)$.

Case 2. When $0 < x < 1$, the inequality (3.3) is equivalent to

$$k \geqq g(x) \quad (0 < x < 1), \tag{3.4}$$

where

$$g(x) = \frac{(x+2)^2 + 3x\sqrt{3(4-x^2)}}{4(7x+2)(x-1)^2(5-2x^2)}[2\sqrt{3}(4-x^2)\sqrt{2x^2+1} - \sqrt{3}(2x^2+1)\sqrt{4-x^2} - (x+2)(5-2x^2)] \quad (0 < x < 1). \tag{3.5}$$

Upon calculating the derivative of $g(x)$ in (3.5), we get

$$g'(x) = \frac{p(x, u, v, w)}{2(7x+2)^2(1-x)^3(2x^2-5)^2\sqrt{2x^2+1}\sqrt{12-3x^2}} \quad (0 < x < 1), \tag{3.6}$$

where

$$\begin{aligned} p(x,u,v,w) = {} & -432ux^9 + (1422u + 228v + 480w)x^8 + (-768v - 324vuw \\ & + 870w + 3150u)x^7 + (-5679u - 288vuw - 3282w - 3696v)x^6 \\ & + (3936v + 1224vuw - 8019u - 6963w)x^5 + (3744vuw + 1041w \\ & + 16305v - 3780u)x^4 + (7032w - 693vuw + 8964u - 915v)x^3 \\ & + (-18960v - 4977vuw + 7320w + 14616u)x^2 + (1440u - 7572v \\ & - 2880vuw + 6864w)x + 1440u - 1680v - 240w - 180vuw \end{aligned} \tag{3.7}$$

and

$$u = \sqrt{3}, \qquad v = \sqrt{2x^2+1} \quad \text{and} \quad w = \sqrt{4-x^2}.$$

For $g'(x)$ and $p(x,u,v,w)$ given by (3.6) and (3.7), respectively, we now solve the equation

$$g'(x) = 0 \quad \text{or} \quad p(x,u,v,w) = 0 \tag{3.8}$$

and consider the following system of nonlinear algebraic equations:

$$\begin{cases} p(x,u,v,w) = 0 \\ u^2 - 3 = 0 \\ v^2 - 2x^2 - 1 = 0 \\ w^2 + x^2 - 4 = 0. \end{cases} \tag{3.9}$$

It is easy to see that the roots of the *second* equation in (3.8) would also provide the solution of the system of nonlinear algebraic equations in (3.9). If we eliminate the ordinals u, v and w by means of Sylvester's resultant (by using Lemma 4), then we get

$$11019960576(7x+2)^8(x+2)^4(x+1)^4(2x^2-5)^8(x-1)^{20}q^2(x) = 0, \tag{3.10}$$

where $q(x)$ is given by

$$\begin{aligned} q(x) = {} & 192x^{10} - 576x^9 + 1600x^8 - 2808x^7 + 2801x^6 - 5832x^5 + 2946x^4 \\ & - 612x^3 + 3833x^2 + 5940x - 2300. \end{aligned} \tag{3.11}$$

It is easily observed that the following algebraic equation:

$$(7x+2)^8(x+2)^4(x+1)^4(2x^2-5)^8(x-1)^{20} = 0 \tag{3.12}$$

has no real root on the interval $(0, 1)$.

The revised sign list of the discriminant sequence of $q(t)$ is given by

$$[1, -1, -1, 1, 1, 1, 1, -1, -1, -1]. \tag{3.13}$$

Consequently, the number of the sign changes of the revised sign list in (3.13) is 3. Thus, by applying Lemma 3, we find that, for $q(x)$ given by (3.11), the following equation:

$$q(x) = 0 \tag{3.14}$$

has 4 distinct real roots. Also, by using the function "realroot()" in *Maple* (Version 9.0) [13, pp. 110–114], we can find that the algebraic equation (3.14) has 4 distinct real roots in the following intervals:

$$\left[\frac{41}{128}, \frac{21}{64}\right], \qquad \left[\frac{181}{128}, \frac{91}{64}\right], \qquad \left[\frac{123}{64}, \frac{247}{128}\right] \quad \text{and} \quad \left[-\frac{101}{128} - \frac{25}{32}\right]. \tag{3.15}$$

Therefore, the algebraic equation (3.14) has only one real root

$$x_0 = 0.3215884740\cdots \tag{3.16}$$

in the open interval $(0, 1)$.

Next, we set

$$v_0 = \sqrt{2x_0^2+1} \quad \text{and} \quad w_0 = \sqrt{4-x_0^2}. \tag{3.17}$$

Then

$$p(x_0, u, v_0, w_0) \approx -404.4633105 < 0. \tag{3.18}$$

Hence x_0 is an extraneous root. It follows that the *second* equation in (3.8) has no real root in the open interval (0, 1), that is, that the *first* equation in (3.8) has no real root in the open interval (0, 1). Furthermore, we have

$$g'\left(\frac{1}{2}\right) = \frac{13525\sqrt{2} - 8762\sqrt{5}}{1089} + \frac{673\sqrt{10} - 2122}{121} < 0. \tag{3.19}$$

Consequently, for any $x \in (0, 1)$, we have

$$g'(x) < 0 \quad (0 < x < 1), \tag{3.20}$$

so the function $g(x)$ given by (3.5) is strictly monotone decreasing on the interval (0, 1). Then

$$\sup_{x\in(0,1)} \{g(x)\} = \lim_{x\to 0+} \{g(x)\} = \frac{3\sqrt{3}}{5} - 1. \tag{3.21}$$

Hence, the best constant k for the inequality (3.4) is given by

$$k = \frac{3\sqrt{3}}{5} - 1,$$

that is, just as asserted by the Theorem, the best constant k for the inequality (1.8) is given by (1.9). The proof of our proposed refinement of the Janous–Gmeiner inequality (1.2) is thus completed.

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